

Heterotic string compactifications on half-flat manifolds II

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ABSTRACT: In this paper, we continue the analysis of heterotic string compactifications on half-flat mirror manifolds by including the 10-dimensional gauge fields. It is argued, that the heterotic Bianchi identity is solved by a variant of the standard embedding. Then, the resulting gauge group in four dimensions is still E_6 despite the fact that the Levi-Civita connection has $SO(6)$ holonomy. We derive the associated four-dimensional effective theories including matter field terms for such compactifications. The results are also extended to more general manifolds with $SU(3)$ structure.

KEYWORDS: Superstrings and Heterotic Strings, Flux compactifications.

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1. Introduction

In recent attempts to stabilise moduli in string compactifications fluxes have played a crucial role. It has also been realised that the notion of flux can be generalised to include geometric fluxes which can be described in terms of manifolds with restricted structure group. In this paper we will concentrate on six-dimensional manifolds with $SU(3)$ structure which are the nearest cousins of Calabi-Yau manifolds. There exists a further generalisation to non-geometric fluxes which are related to backgrounds with $SU(3) \times SU(3)$ structure, but in this paper we will stay in the realm of geometric compactifications.

In the context of heterotic string compactifications, manifolds with $SU(3)$ structure play an important role. Soon after Calabi-Yau compactifications were proposed in ref. [1] it was realised that in the presence of H -flux, the supersymmetric ground state of the heterotic string corresponds to an internal manifold which is complex, but non-Kähler [2, 3]. More recently, in refs. [4–17] such compactifications were classified in terms of $SU(3)$ structures

which is the natural way to approach this problem. It was argued in refs. [7, 8, 11] that in this way a superpotential is generated and some of the Calabi-Yau moduli get fixed.

This mechanism for moduli stabilisation is one of the most attractive phenomenological features of $SU(3)$ structure compactifications, particularly in the context of the heterotic string, where only one type of conventional flux, NS-NS flux, is available. For Calabi-Yau compactifications, this stabilises the complex structure moduli [18] but not the Kähler moduli (and the dilaton) which remain flat directions. The only known way to generate a perturbative superpotential for the Kähler moduli is indeed to use manifolds with $SU(3)$ structure. In this paper we shall restrict our attention to a class of manifolds with $SU(3)$ structure — termed half-flat manifolds [19] — which first appeared in string compactifications in the context of mirror symmetry in the presence of NS-NS fluxes [20, 21]. In order to distinguish the specific type of manifolds discussed in [20, 21] from the more general family of half-flat manifolds we will denote them as *half-flat mirror manifolds*. A more explicit explanation of what we understand by these manifolds can be found at the beginning of section 3.1. In ref. [22] the superpotential for such manifolds, was derived and subsequently analysed in ref. [23]. However, the analysis was restricted to the gravitational sector (zeroth order in α') and the gauge/matter sector was not addressed in detail. A detailed analysis turned out to be quite involved due to the large number of terms which appear in the reduction of a 10-dimensional gauge theory to four dimensions.

In this paper, we will show how to overcome these difficulties, using the heterotic Gukov formula for the superpotential (for the original version of the formula in the context of type IIA and M theory see refs. [24, 25]). Based on this approach we will explicitly calculate the four-dimensional effective theory including the gauge field sector for heterotic compactifications on half-flat mirror manifolds and their generalisations. This result will allow us to address a number of questions which have been the main motivation behind this work. What is the four-dimensional low-energy gauge group for such non Calabi-Yau compactifications? What is the four-dimensional (gauge matter) particle spectrum? Do any of these four-dimensional gauge matter fields pick up masses from the (geometrical) fluxes? Is the low-energy gauge group spontaneously broken?

Let us explain in more detail how we proceed in deriving the four-dimensional effective theory. One of the main obstacles to overcome in heterotic string compactifications is to find a solution to the Bianchi identity for the NS-NS field strength H . Since our knowledge about manifolds with $SU(3)$ structure is fairly limited explicitly constructing bundles over such spaces is an ambitious task (for recent developments see refs. [26, 27]). Nevertheless we always have the standard embedding at our disposal where the background gauge field is set equal to the spin connection. It represents the simplest solution to the Bianchi identity and will be adopted in this paper. To determine the expansion of the 10-dimensional fields into low-energy modes we will be guided by an "perturbative principle" which has been proposed and successfully applied to type II string theory in ref. [20, 21] and has been shown in ref. [22] to lead to a consistent description of the gravitational sector in heterotic theories. The basic assumption underlying this principle is that half-flat mirror manifolds (and their generalisations) can be considered as "perturbations" of Calabi-Yau manifolds and, hence, that their low-energy spectrum is identical to the one of the associated Calabi-Yau

manifolds. We will show that this principle leads to a consistent description of the heterotic gauge field sector. In particular, we will show that the low-energy gauge group remains E_6 in agreement with the perturbative approach. This conclusion may be surprising since the Levi-Civita connection for manifolds with $SU(3)$ structure has, in general, a holonomy group $SO(6) \simeq SU(4)$ which suggests the low-energy gauge group $SO(10)$. This is, in fact, what has been proposed in ref. [22]. Here we will show that E_6 is the correct low-energy gauge group.

Having decided upon the expansion of 10-dimensional fields into low-energy modes we will use the heterotic Gukov formula for the superpotential (gravitino mass)

$$e^{K/2}W = \frac{e^\phi}{\sqrt{2}\mathcal{K}||\Omega||} \int \Omega \wedge (H + idJ) , \tag{1.1}$$

which was derived from first principles in ref. [22]. It provides a way of computing the Kähler potential K and superpotential W of the low-energy theory in terms of the NS-NS flux H and the forms Ω and J which characterise the $SU(3)$ structure. For half-flat mirror manifolds, both H and dJ are non-zero at zeroth order in α' and this leads to a superpotential which is linear in the Kähler moduli [22]. At first order in α' the above formula receives a contribution from the Chern-Simons correction to H . Given the expansion of the gauge fields we can explicitly compute the Chern-Simons term and integrate to obtain the superpotential including matter field terms.

It has been known since the early work in ref. [28] that the definition of the Kähler moduli in terms of the 10-dimensional fields is modified at first order in α' by a certain combination of the matter fields. This aspect of compactifications with matter fields often significantly complicates the task of finding the correct definitions of the low-energy fields. In our context this problem can be quite elegantly dealt with. Due to the existence of a Kähler-moduli dependent superpotential at zeroth order in α' one expects the matter field corrections to appear at order α' in eq. (1.1). This is indeed what we will find. The correct definition of the moduli can then be read off from these additional terms. Based on these ideas we will carry out the full reduction for half-flat mirror manifolds [20, 22] and then extend the results to the generalised half-flat manifolds introduced in refs. [29, 30].

The paper is organised as follows. We start with a review of heterotic Calabi-Yau compactifications based on the standard embedding. This is mostly to fix conventions and as a reference point for the more involved calculation in the half-flat case. Then, in section 3, we perform the analogous analysis for half-flat mirror manifolds. In section 3.2, we first discuss the Bianchi identity, in section 3.3 we compute the gravitino mass and, finally, in section 3.4, we find the associated four-dimensional effective theory. For the correct interpretation of the result it will prove useful to include H -flux which we will do in section 3.5. Then, in section 4, we extend the results of section 3 to generalised half-flat manifolds. Finally, in section 5, we conclude and present future directions for research. Formulae and conventions for Calabi-Yau manifolds and the group E_8 have been collected in the appendix.

2. Warm up: Calabi-Yau compactifications of the heterotic string

In this section we perform the compactification of the heterotic string on Calabi-Yau manifolds using the standard embedding. Since this material is well-known we will keep the discussion brief and we refer the reader to the standard textbooks, for example [31], for a more detailed treatment. However, in view of our earlier discussion, we will perform the computation in the gauge matter sector in an unusual way, using the Gukov formula (1.1).

2.1 The spectrum in 10 and 4 dimensions

We start with the low energy action of the heterotic string in 10 dimensions which is given by supergravity coupled to a super-Yang-Mills theory with gauge group $E_8 \times E_8$.¹ The bosonic spectrum — which is what we are mostly interested in — is given by the graviton G_{MN} , the antisymmetric tensor field B_{MN} and the dilaton ϕ in the gravity sector and by the gauge fields A_M^I , where I is an adjoint index of $E_8 \times E_8$ and runs from $1, \dots, 496$. The action for these fields is given by

$$S = -\frac{1}{2} \int e^{-2\phi} \left[R * \mathbf{1} - 4d\phi \wedge *d\phi + \frac{1}{2} H \wedge *H + \alpha' \left(\text{Tr} F \wedge *F - \text{tr} \tilde{R}^2 \right) \right], \quad (2.1)$$

where $\text{tr} \tilde{R}^2$ stands for the Gauss-Bonnet combination. Here we have kept the dependence on α' which is going to be a useful expansion parameter. We will neglect any terms at order α'^2 or higher. The field strengths F and H are defined by

$$F = dA + A \wedge A, \quad (2.2)$$

and

$$H = dB + \alpha' (\omega_L - \omega_{\text{YM}}). \quad (2.3)$$

Here ω_L and ω_{YM} are the Chern-Simons three-forms

$$\omega_L = \text{tr} \left(\tilde{R} \wedge \tilde{w} - \frac{1}{3} \tilde{w} \wedge \tilde{w} \wedge \tilde{w} \right), \quad (2.4)$$

$$\omega_{\text{YM}} = \text{Tr} \left(F \wedge A - \frac{1}{3} A \wedge A \wedge A \right). \quad (2.5)$$

The modified spin connection one-form, \tilde{w} , is given in terms of the Levi-Civita connection, w , by [32]

$$\tilde{w}_{MN}{}^P = w_{MN}{}^P - \frac{1}{2} H_{MN}{}^P, \quad (2.6)$$

and all curvature tensors denoted by \tilde{R} are computed in terms of this modified connection. Finally, the symbol Tr above denotes $1/30$ of the trace in the adjoint of $E_8 \times E_8$.

Taking the exterior derivative of eq. (2.3) one obtains the well-known Bianchi identity

$$dH = \alpha' (\text{tr} \tilde{R} \wedge \tilde{R} - \text{Tr} F \wedge F). \quad (2.7)$$

¹In this paper we focus on $E_8 \times E_8$ and do not discuss the $\text{SO}(32)$ case.

The simplest solution to this equation — which we will also adopt in this paper — is the so called standard embedding where the background gauge field (for one of the E_8 group factors) is set equal to the spin connection \tilde{w} . In the absence of H -fluxes the connection has $SU(3)$ holonomy and this choice of background breaks the gauge group to $E_6 \times E_8$. Here and in the rest of the paper we shall always have a "hidden sector" E_8 gauge factor which we will often omit from the discussion. The gauge matter fields which reside in four-dimensional chiral multiplets arise from the internal components of the gauge fields and consist of $h^{2,1}$ $\mathbf{27}$ -plets and $h^{1,1}$ $\overline{\mathbf{27}}$ -plets where $h^{1,1}$ and $h^{2,1}$ denote the Hodge numbers of the Calabi-Yau manifold. Therefore the net number of families is given by $|h^{1,1} - h^{2,1}| = |\chi|/2$.

In addition to the fields discussed above there are a number of gravitational fields. Apart from the four-dimensional metric tensor in the gravity multiplet we have $h^{1,1}$ (complexified) Kähler moduli, t^i , and $h^{2,1}$ complex structure moduli, z^a , as well as the axio-dilaton s , all of them in four-dimensional chiral multiplets. Finally we have the so called "bundle moduli" which parameterise deformations of the gauge bundle. In this paper we will not be concerned with this last class of fields and we will ignore them.

2.2 Four-dimensional effective action

Let us move on to the four-dimensional action for the fields described above. It will be useful to organise the discussion according to the order in α' at which different terms appear. Let us start with order zero. At this order, only the compactification of ten dimensional supergravity contributes and leads to the four-dimensional supergravity sector coupled to the moduli fields. The action is given by²

$$S_{0,\text{kinetic}} = - \int \left[\frac{1}{2} R * \mathbf{1} + d\phi \wedge *d\phi + \frac{1}{4} e^{4\phi} da \wedge *da + g_{ij} dt^i \wedge *d\bar{t}^j + g_{a\bar{b}} dz^a \wedge *d\bar{z}^b \right]$$

where t^i denote the complexified Kähler moduli. They are obtained by expanding the complexified Kähler form $B + iJ$ into two-forms ω_i ,

$$B + iJ = (b^i + iv^i)\omega_i \equiv t^i \omega_i, \quad (2.8)$$

with $i, j, \dots = 1, \dots, h^{1,1}$. Further, z^a denote the complex structure moduli introduced in eq. (A.5), ϕ is the four-dimensional dilaton and its partner a , the universal axion, is the dual of the four-dimensional tensor field $B_{\mu\nu}$. The kinetic terms in (2.8) can be obtained from the usual zeroth order Kähler potential, K_0 , given by

$$K_0 = K_K + K_{\text{cs}} + K_s, \quad (2.9)$$

with

$$K_{\text{cs}} = - \ln (\mathcal{K} |\Omega|^2), \quad (2.10)$$

$$K_K = - \ln \mathcal{K}, \quad (2.11)$$

$$K_s = - \log(i(\bar{s} - s)), \quad (2.12)$$

²For our conventions, see appendix A.

and

$$s = a + ie^{-2\phi} , \quad (2.13)$$

Here, \mathcal{K} denotes the Calabi-Yau volume (A.12) and Ω is the holomorphic (3,0) form. Explicit expressions for K_K and K_{cs} are given in appendix A.2. At zeroth order in α' , the superpotential W_0 vanishes and, hence, the above Kähler potential completely specifies the theory at this order.

Let us now discuss the ten-dimensional gauge fields and their descendants in four dimensions which appear at first order in α' . For the standard embedding case the massless matter fields can be obtained from expanding the internal gauge-fields in (0,1) harmonic forms with values in the holomorphic and anti-holomorphic tangent bundle. On a Calabi-Yau manifold these spaces are known to be isomorphic to the cohomology groups $H^{2,1}(X)$ and $H^{1,1}(X)$ and we can, therefore, write

$$A_{\bar{\alpha}} = A_{\bar{\alpha}}^{(0)} + A_{\bar{\alpha}}^{(1)} \quad (2.14)$$

$$A_{\bar{\alpha}}^{(1)} = \|\Omega\|^{-1/3} (\omega_i)_{\bar{\alpha}}^{\bar{\beta}} e_{\bar{\beta}}^{\bar{\gamma}} \bar{T}_{\bar{\beta}\bar{P}} C^{i\bar{P}} + \|\Omega\|^{1/3} (\eta_a)_{\bar{\alpha}}^{\beta} e_{\beta}^{\bar{\gamma}} T_{\bar{\beta}P} D^{aP} . \quad (2.15)$$

Here $C^{i\bar{P}}$ and D^{aP} denote the matter fields in the $\overline{27}$ and 27 respectively, ω_i are the harmonic (1,1) forms introduced earlier and the rank two symmetric tensors η_a are defined in terms of the (2,1) forms, see eq. (A.15). By $A^{(0)}$ we have denoted the background gauge field and $A^{(1)}$ contains the matter field fluctuations around this background. Having chosen the standard embedding, the background is set equal to the Calabi-Yau spin-connection, that is

$$A_{\bar{\alpha}}^{(0)} = w_{\bar{\alpha}\beta\bar{\gamma}} S^{\beta\bar{\gamma}} , \quad (2.16)$$

where $S^{\beta\bar{\gamma}}$ are the generators of $SU(3) \subset E_8$, defined in appendix A.3. Note that the indices β and $\bar{\gamma}$ should in principle be tangent space indices. We have glossed over this subtlety in the above formula in order not to overload the notation. In (2.14) however, the nature of indices is important, especially when taking derivatives, and therefore we have explicitly included the vielbeins to convert between curved and tangent space indices. Note the factors $\|\Omega\|^{\pm 1/3}$ in the expansion (2.15) which correspond to a particular definition of the gauge matter fields $C^{i\bar{P}}$ and D^{aP} . As we shall see below these factors are required in order to make the superpotential holomorphic. Note that $\|\Omega\|$ does not depend on the Calabi-Yau coordinates and, therefore, these additional factors do not complicate the calculation.

The kinetic terms for the gauge matter fields can be easily obtained by calculating the field strength (specifically the components with one internal and one external leg) associated to eq. (2.15) and inserting the result into the Yang-Mills part of the 10-dimensional action (2.1). One finds

$$S_{1,\text{kinetic}} = -\alpha' \int \left[4g_{ij} \|\Omega\|^{-2/3} dC_P^i \wedge *d\bar{C}^{jP} + g_{a\bar{b}} \|\Omega\|^{2/3} d\bar{D}^{aP} \wedge *dD_P^{\bar{b}} \right] . \quad (2.17)$$

These kinetic terms can be obtained from the matter field Kähler potential

$$K_1 = 4\alpha' g_{ij} \|\Omega\|^{-2/3} C_P^i \bar{C}^{jP} + \alpha' g_{a\bar{b}} \|\Omega\|^{2/3} D^{aP} \bar{D}_P^{\bar{b}} , \quad (2.18)$$

which should be added to the zeroth order result (2.9). Up to constant normalisations this is precisely the Kähler potential computed in ref. [33], which confirms the expansion (2.15).

The scalar potential for the gauge matter fields can be calculated by computing the purely internal components of the field strength associated to eqs. (2.14), (2.15) and inserting the result into the Yang-Mills part of the 10-dimensional action (2.1). One may then attempt to read off the superpotential and the D -terms from the result. However, as discussed earlier, there is a simpler and cleaner way of deriving the superpotential based on the heterotic Gukov formula (1.1). Let us follow this route and see how the known result for the matter field superpotential can be reproduced.

In the case presently under discussion, that is Calabi-Yau compactifications in the absence of H -flux, no superpotential arises at zeroth order in α' as is clear from the Gukov formula (1.1). However, at first order in α' , H can still pick up a purely internal component due to the Chern-Simons term in eq. (2.7). Therefore, using eqs. (2.3), (2.4) and (2.5) with the gauge field Ansatz (2.14) inserted, the Gukov formula (1.1) can be written as

$$W = \alpha' \int \Omega \wedge (\omega_L - \omega_{\text{YM}}) = -\alpha' \int \Omega \wedge \text{Tr} \left(F \wedge A - \frac{1}{3} A \wedge A \wedge A \right)^{(1)}. \quad (2.19)$$

In this equation, the pure Yang-Mills background contribution due to $A^{(0)}$ and its field strength

$$F^{(0)} = dA^{(0)} + A^{(0)} \wedge A^{(0)}. \quad (2.20)$$

is canceled by the Lorentz Chern-Simons term by virtue of the standard embedding. It should, therefore, be omitted from the expression on the r.h.s. of eq. (2.19) which is indicated by the superscript (1). Hence, only terms at least linear in $A^{(1)}$ or its field strength contribute. In addition, due to the presence of Ω in eq. (2.19) only the (0,3) piece of the Chern-Simons term is relevant. Therefore only the (0,2) part of F , $F_{(0,2)}$, enters the calculation. Since we have expanded the gauge fields in (0,1) harmonic forms with values in the (anti)holomorphic tangent bundle the part of $F_{(0,2)}$ linear in $A^{(1)}$ vanishes. Let us see more explicitly how this works. The terms which are linear in the matter fields can schematically be written as

$$F^{(1)} = dA^{(1)} + 2[A^{(0)}, A^{(1)}]. \quad (2.21)$$

In the first term the derivative can act on the forms ω_i and η_a or on the vielbeins. When the derivatives act on the forms then, due to antisymmetrisation, we end up with exterior derivatives, which vanish in the Calabi-Yau case. The remainder takes the form

$$\begin{aligned} (F^{(1)})_{\bar{\alpha}\bar{\beta}} &= \|\Omega\|^{1/3} (\eta_a)_{\bar{\alpha}}{}^\gamma \left[\nabla_{\bar{\beta}} e_{\gamma}{}^{\underline{\gamma}} T_{\underline{\gamma}P} + w_{\bar{\beta}\bar{\gamma}} [T_{\underline{\delta}P}, S^{\beta\bar{\gamma}}] e_{\gamma}{}^{\underline{\delta}} \right] - (\bar{\alpha} \leftrightarrow \bar{\beta}) + \dots \\ &= \|\Omega\|^{1/3} (\eta_a)_{\bar{\alpha}}{}^\gamma \left[\nabla_{\bar{\beta}} e_{\gamma}{}^{\underline{\gamma}} - w_{\bar{\beta}\bar{\delta}} e_{\gamma}{}^{\underline{\delta}} \right] T_{\underline{\gamma}P} - (\bar{\alpha} \leftrightarrow \bar{\beta}) + \dots, \end{aligned} \quad (2.22)$$

where we have focused on terms containing η_a and the dots stand for analogous terms containing ω_i . In the last equality we have used the commutator (A.26). Note that the second term in this commutation relation does not contribute as the spin connection is a $SU(3)$ Lie algebra valued one-form and therefore its contraction with the metric vanishes.

In the bracket we recognise the defining equation for the spin connection in terms of the vielbein and, hence, the above terms vanish. The same conclusion holds for the terms containing ω_i and, hence, $F^{(1)}$ is zero. Thus we have indeed shown that (0,2) part of the field strength originates from the commutator term, that is

$$F_{(0,2)} = A^{(1)} \wedge A^{(1)} , \quad (2.23)$$

or in components

$$F_{\bar{\alpha}\bar{\beta}} = \left[\|\Omega\|^{-1/3} (\omega_i)_{\bar{\alpha}}^{\bar{\gamma}} \bar{T}_{\bar{\gamma}\bar{P}} C^{i\bar{P}} + \|\Omega\|^{1/3} (\eta_a)_{\bar{\alpha}}^{\bar{\gamma}} T_{\gamma P} D^{aP} , \|\Omega\|^{-1/3} (\omega_j)_{\bar{\beta}}^{\bar{\delta}} \bar{T}_{\bar{\delta}\bar{R}} C^{j\bar{R}} + \|\Omega\|^{1/3} (\eta_b)_{\bar{\beta}}^{\bar{\delta}} T_{\delta R} D^{bR} \right] . \quad (2.24)$$

Based on this result, let us now perform a similar analysis for the full combination of Chern-Simons terms in (2.19). We have already mentioned that the pure background part in this combination cancels between the gravity and gauge field terms due to the standard embedding. Linear terms in $A^{(1)}$ cannot be present as they would lead to gauge non-invariant terms in the superpotential. Hence, we are left with quadratic and cubic terms in $A^{(1)}$. However, using (2.23), it is easy to see that the terms quadratic in $A^{(1)}$ cancel in eq. (2.19). Therefore we can write the superpotential for the charged fields as

$$W = -\alpha'^2 \int \Omega \wedge \text{Tr} \left(A^{(1)} \wedge A^{(1)} \wedge A^{(1)} \right) . \quad (2.25)$$

Substituting the expression (2.15) for $A^{(1)}$ we obtain the final result for the order α' superpotential, W_1 , which reads³

$$W_1 = -\frac{1}{3} \alpha' \left[\bar{j}_{\bar{P}\bar{R}\bar{S}} \bar{\mathcal{K}}_{ijk} C^{i\bar{P}} C^{j\bar{R}} C^{k\bar{S}} + j_{PRS} \tilde{\mathcal{K}}_{abc} D^{aP} D^{bR} D^{cS} \right] . \quad (2.26)$$

Here we have used the trace relation (A.30) and \mathcal{K}_{ijk} and $\tilde{\mathcal{K}}_{abc}$ are the triple intersection numbers defined in eqs. (A.18) and (A.21). This is the well-known cubic superpotential which we have derived from the heterotic Gukov formula (1.1).

In order to compute the $N = 1$ supergravity potential we also need the D -terms. Given that we know the matter field content and the Kähler potential, these can, of course, be calculated purely from $N = 1$ supergravity [35]

$$D^x = G_I (T^x)^I J \xi^J . \quad (2.27)$$

Here, G_I are the derivatives of the supergravity G -function, $G = K + \ln|W|^2$, with respect to the matter fields ϕ^I and T^x denote the gauge group generators in the representation in which ξ_I transforms. However, as a useful consistency check, the D -terms can also be directly obtained from 10 dimensions using the formula [36]

$$D^x = \int F^x \wedge *J , \quad (2.28)$$

³In the context of heterotic M-theory a similar method for deriving this superpotential was recently used in [34].

which is similar to the Gukov formula (1.1) for the superpotential. Here x is an E_6 adjoint index. Since J is a $(1, 1)$ -form we need the $(1, 1)$ components of F to evaluate the expression on the r.h.s. . They can be calculated similar to the $(0, 2)$ component of F above, the main difference being that antisymmetrisation of the indices does not lead to exterior derivatives so terms with derivatives acting on forms no longer vanish. Instead, we have

$$F_{\alpha\bar{\beta}} = F_{\alpha\bar{\beta}}^{(0)} + \|\Omega\|^{-1/3} \nabla_{\alpha}(\omega_i)_{\bar{\beta}} \bar{T}_{\bar{\gamma}\bar{P}}^{\bar{\gamma}} C^{i\bar{P}} + \|\Omega\|^{1/3} \nabla_{\alpha}(\eta_a)_{\bar{\beta}} \gamma T_{\gamma P} D^{aP} - \|\Omega\|^{-1/3} \nabla_{\bar{\beta}}(\omega_i)_{\alpha} \gamma T_{\gamma P} \bar{C}^{iP} - \|\Omega\|^{1/3} \nabla_{\bar{\beta}}(\bar{\eta}_{\bar{a}})_{\alpha} \bar{\gamma} \bar{T}_{\bar{\gamma}\bar{P}} \bar{D}^{\bar{a}\bar{P}} + [A^{(1)}, A^{(1)}] . \quad (2.29)$$

Contributions to the D -terms can only arise from terms which involve E_6 generators and, hence, recalling the commutation relations (A.25), only from the last term in the above expression. Performing the integral in (2.28) we obtain

$$D^x = 4\|\Omega\|^{-2/3} g_{ij} C^{i\bar{P}} \bar{C}^{jR} k_{\bar{P}\bar{R}}^x - \|\Omega\|^{2/3} g_{\bar{a}\bar{b}} D^{aP} \bar{D}^{\bar{b}\bar{R}} k_{P\bar{R}}^x , \quad (2.30)$$

which is precisely what one obtains from the supergravity formula (2.27) applied to (2.18). This ends our review of heterotic Calabi-Yau compactifications with standard embedding.

3. Compactification on half-flat mirror manifolds

In this section, we will compactify the heterotic string, including its gauge field sector, on half-flat mirror manifolds [20], following the strategy outlined in the previous section. Due to the lack of explicit constructions for these manifolds, a rigorous derivation is not really possible. Instead we will adopt a “perturbative approach”, as in ref. [20, 21]. In general, a Calabi-Yau manifold, Y , has a mirror dual, X and a half-flat mirror dual, X' obtained when NS-NS flux is added on Y . In ref. [20] it was argued that in a certain region of the moduli space one can view the manifold X' as a small variation of the manifold X . We shall precisely work under these assumptions and we consequently denote the manifold X as the Calabi-Yau manifold related or associated to the half-flat mirror manifold X' . Many of the standard Calabi-Yau methods on X can then be transferred to the half-flat mirror manifold X' . This approach has been successfully applied to type II mirror symmetry [20] and to the gravitation sector of the heterotic string [22].

Let us be more explicit about this approximation and point out some of its consequences. As it will become clear in the next section we will introduce parameters e_i to quantify the departure from ordinary Calabi-Yau manifolds. Therefore one of the main assumption is that these parameters are small and terms which contain more than two powers of e_i will be neglected. By the perturbative approach we will still consider a ten-dimensional background metric which is a product of a four-dimensional Minkowski space and the metric on the internal manifold. For this to be a solution of the ten-dimensional supersymmetry equations, the manifold with $SU(3)$ structure has to obey certain conditions [5, 6, 12, 15]. We do not impose these conditions on the compactification manifolds in the first place as our purpose here is only to obtain a low energy effective action. The condition we should however require for a consistent result, is that the (supersymmetric) minima of the (super)potential we find here should be at points in the moduli space where

the ten-dimensional equations are indeed satisfied. We will deal with these conditions elsewhere [37].

After a brief review of manifolds with $SU(3)$ structure and half-flat mirror manifolds in the next sub-section, we will start by finding a solution to the Bianchi identity (2.7) for half-flat mirror manifolds which is analogous to the standard embedding. Then, we will evaluate the heterotic Gukov formula (1.1) for half-flat mirror manifolds. In the pure Calabi-Yau case, evaluating this formula has given us information merely about the matter field superpotential but not the Kähler potential, despite the explicit Kähler potential dependence in eq. (1.1). The reason is that the superpotential in the standard Calabi-Yau case is of order α' and, hence, calculating the matter field Kähler potential (which is also of order α') requires evaluating eq. (1.1) at order α'^2 . Terms at this order are beyond the scope of our calculation and, therefore, we resorted to a standard reduction of the Yang-Mills action to determine the Kähler potential from the matter field kinetic terms. For half-flat mirror manifolds, on the other hand, the torsion flux generates a superpotential at zeroth order in α' . As a result, the Gukov formula at order α' will provide us with information about both the superpotential and the matter field Kähler potential. In addition, we will be able to infer another crucial piece of information, namely the correct definition of the Kähler moduli fields, which receive order α' matter field corrections, as is well-known for the Calabi-Yau case [28]. Generalising our set-up further by adding H -flux then allows us to find an analogous correction to the definition of the complex structure moduli. As we will see, this provides us with sufficient information to completely determine the four-dimensional gauge matter field action at order α' .

3.1 Half-flat $SU(3)$ structure manifolds

Before we proceed with the computation, we review some of the required properties of manifolds with $SU(3)$ structure and the specific sub-class of half-flat (mirror) manifolds (for a more formal description of manifolds with $SU(3)$ structure see for example [19]). Manifolds with $SU(3)$ structure are almost complex manifolds for which the structure group of the frame bundle reduces to $SU(3)$. They can be described in terms of an invariant two-form J (the fundamental form) and an invariant three-form Ω which is of type $(3,0)$ with respect to the almost complex structure. Manifolds with $SU(3)$ structure can be classified according to their intrinsic torsion τ , which is associated to the connection which preserves the structure (that is, which annihilates the forms J and Ω). The intrinsic torsion is a one-form taking values in $\mathfrak{su}(3)^\perp$ where

$$\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp . \tag{3.1}$$

Here $\mathfrak{so}(6) \sim \mathbf{15}$ and $\mathfrak{su}(3) \sim \mathbf{8}$ denote the Lie algebras of $SO(6)$ and $SU(3)$, respectively, and $\mathfrak{su}(3)^\perp \sim \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$ is the part perpendicular to $\mathfrak{su}(3)$. Unlike for Calabi-Yau manifolds, the forms J and Ω are no longer closed and the expressions for dJ and $d\Omega$ can be used to read off the intrinsic torsion τ and, hence, to characterise the manifold. Half-flat manifolds are formally defined by imposing additional restrictions on dJ and $d\Omega$, namely

$$d(J \wedge J) = d\text{Im}\Omega = 0 . \tag{3.2}$$

These remove half of the torsion components which is why these manifolds are sometimes also called half-integrable.

For practical purposes it is more useful to define a class of half-flat manifolds starting from related Calabi-Yau manifolds [20]. As mentioned earlier, this perturbative approach has the advantage of providing a fairly explicitly framework for calculations with many of the standard Calabi-Yau techniques applicable. One starts by postulating the existence of a set of two-forms ω_i and a symplectic set of three-forms (α_A, β^B) , where $(\alpha_A) = (\alpha_0, \alpha_a)$ and $(\alpha^B) = (\alpha^0, \alpha^b)$. They are of course the analogue of the harmonic two and three forms on a Calabi-Yau manifold and still satisfy the standard normalisation relations (A.1) and (A.2). However, they are no longer closed but instead satisfy the identities

$$\begin{aligned} d\omega_i &= e_i \beta^0, & d\tilde{\omega}^i &= 0, \\ d\alpha_0 &= e_i \tilde{\omega}^i, & d\alpha_a &= d\beta^A = 0, \end{aligned} \tag{3.3}$$

where e_i are torsion flux parameters. Apart from this modification, the properties of Calabi-Yau manifolds listed in appendix A.2 are assumed to remain valid. In particular, the moduli are defined by expanding the SU(3) invariant forms J and Ω into the above sets of forms, as in eqs. (A.3) and (A.4). Then, it is easy to see that the first of the half-flat conditions (3.2) is implied by the primitivity of the three forms (α_A, β^A) on a Calabi-Yau manifold. The second condition is satisfied because the standard choice $\mathcal{Z}^0 = 1$ implies that $\text{Im}\Omega$ does not contain α_0 which is the only non-closed three-form. We will refer to half-flat manifolds with the above set of forms and properties as half-flat mirror manifolds, due to their appearance in the context of type II mirror symmetry with NS-NS flux [20].

The heterotic string on such half-flat mirror manifolds at zeroth order in α' has been discussed in ref. [22] and the results can be easily summarised. Since most of the standard Calabi-Yau relations still hold it follows that the moduli Kähler potential (2.9) remains unchanged from the Calabi-Yau case. The only modification at zeroth order in α' is the appearance of a superpotential [22]

$$W_0 = e_i t^i, \tag{3.4}$$

for the Kähler moduli t^i , as can be easily seen from eq. (1.1). For later purposes it will be useful to explicitly derive the components of the intrinsic torsion from the relations (3.3). It is not hard to show [20] that

$$\begin{aligned} \tau_{\alpha\beta}{}^{\tilde{\gamma}} &= \frac{1}{4\|\Omega\|^2} (e_i \tilde{\omega}^i)_{\alpha\beta\tilde{\alpha}\tilde{\beta}} \Omega^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}, \\ \tau_{\alpha\beta}{}^{\gamma} &= -\frac{i}{2} (e_i v^i) (\beta^0)_{\alpha\beta}{}^{\gamma}, \end{aligned} \tag{3.5}$$

with components other than the complex conjugate of the above being zero. It is important to note that the first two indices of the torsion are of the same complex type. Further, primitivity of β^0 implies the contraction of the torsion tensor with J vanishes, that is

$$\tau_{\text{mnp}} J^{\text{np}} = 0. \tag{3.6}$$

Hence, the torsion tensor has no components in the singlet part of $\mathfrak{su}(3)^\perp \sim \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$. The contorsion tensor, κ , which we shall also need later on, can be written in terms of the torsion tensor as

$$\kappa_{mnp} = \tau_{mnp} + \tau_{pnm} + \tau_{pnm} . \tag{3.7}$$

Eqs. (3.5) and (3.6) imply that the singlet part of κ also vanishes. It is also worth pointing out that the internal components of the field strength H are non-zero for half-flat mirror manifolds, even in the absence of genuine NS-NS flux. This happens because the forms ω_i are no longer closed and, hence, taking the exterior derivative of the B field Ansatz (2.8) together with eq. (3.3) one finds, apart from the usual terms involving four-dimensional derivatives, that

$$H = e_i b^i \beta^0 . \tag{3.8}$$

3.2 Solving the Bianchi identity

In order to apply the Gukov formula (1.1) we need to compute the field strength H from its definition (2.3). This, in turn, requires finding a background gauge field configuration which satisfies the Bianchi identity (2.7). Here, we would like to discuss the simplest possibility, that is a gauge field background obtained from a standard embedding. However, things are not quite so straightforward as we have various connections available to set the gauge field equal to. The most immediate choice seems to be to set the gauge field equal to the Levi-Civita connection, w , of the half-flat mirror manifold. There are two obvious problems with such a choice. Firstly, it is the modified connection $\tilde{w} = w - H/2$ which enters the curvature term in the Bianchi identity. While this made no difference in the Calabi-Yau case, it does here since, as we have seen in eq. (3.8), the internal part of H is non-vanishing. Therefore, setting the gauge field equal to w means the Bianchi identity is not strictly satisfied. Secondly, and perhaps more importantly, w (and presumably \tilde{w} as well) has holonomy $\text{SO}(6)$, leading to a gauge symmetry breaking to $\text{SO}(10)$ rather than E_6 . In fact, such a breaking to $\text{SO}(10)$ was predicted in ref. [22]. However, the perturbative approach dictates that low-energy modes should be the same as for the Calabi-Yau case and that, consequently, the gauge group should be E_6 . Such a breaking can be realised with the torsion connection $w^{(T)}$ which has $\text{SU}(3)$ holonomy. Schematically, it is related to \tilde{w} by

$$\tilde{w}_m = w_m^{(T)} + \Upsilon_m^\parallel + \Upsilon_m^\perp , \tag{3.9}$$

where $\Upsilon_m^\parallel \in \mathfrak{so}(3)$ and $\Upsilon_m^\perp \in \mathfrak{so}(3)^\perp$, and the tensor Υ will be explicitly determined below. This way of splitting up the connection \tilde{w} is in line with the possible steps for gauge symmetry breaking. Specifically, the first two terms preserve an E_6 gauge group, while the further breaking to $\text{SO}(10)$ is only due to Υ_m^\perp . We will show that this part can be absorbed into a re-definition of the matter gauge fields C and D . The difference between choosing \tilde{w} and $w_m^{(T)} + \Upsilon_m^\parallel = \tilde{w} - \Upsilon_m^\perp$ therefore amounts to a shift of the four-dimensional fields. Generally, one would expect that a sensible choice of connection leads to a low-energy supersymmetric vacuum with vanishing matter fields, $C = D = 0$, and unbroken gauge symmetry, while less suitable choices for the background may lead to non-vanishing VEVs for C and D and symmetry breaking (or restoration). In keeping with the perturbative

approach we will set the gauge field equal to $w_m^{(T)} + \Upsilon_m^\parallel$, so that the low-energy gauge group at this stage is E_6 . As we will show, for this choice of background there is indeed always a vacuum with $C = D = 0$ and E_6 unbroken.

Let us try to make the above discussion more precise. We start by splitting Lie-algebra valued forms into $\mathfrak{su}(3)$ and $\mathfrak{su}(3)^\perp$ parts which we denote by superscripts \parallel and \perp , respectively. The torsion connection $w^{(T)}$ takes of course values in $\mathfrak{su}(3)$ while, for the Levi-Civita connection, we can write⁴

$$w_m = w_m^{(T)} - \kappa_m^\parallel - \kappa_m^\perp. \quad (3.10)$$

Similarly, we can also think of H as an $\mathfrak{so}(6)$ valued one-form which can be decomposed as

$$H_m = H_m^\parallel + H_m^\perp. \quad (3.11)$$

As the three forms β^A are primitive, the explicit form (3.8) of H implies that the singlet part in H^\perp is also zero.

Since H and κ appear on the same footing in the modified connection \tilde{w} in eq. (2.6) it is useful to introduce the notation

$$\Upsilon_{mnp} = - \left(\kappa_{mnp} + \frac{1}{2} H_{mnp} \right), \quad (3.12)$$

which leads to eq. (3.9). Given the expressions (3.8) for H and for the (con)torsion tensor for a half-flat mirror manifold, eqs. (3.5) and (3.7), the tensor Υ^\parallel takes the form

$$(\Upsilon_\alpha^\parallel)_{\alpha\bar{\beta}} = -\frac{1}{2} (e_i t^i) (\beta^0)_{\alpha\alpha\bar{\beta}}, \quad (3.13)$$

while, using eq. (A.9), the orthogonal component can be written as

$$\begin{aligned} (\Upsilon_\alpha^\perp)_{\alpha\beta} \bar{\Omega}^{\alpha\beta\gamma} &= -i \frac{e_i t^i}{\mathcal{K}} g^{a\bar{b}} K_{\bar{b}}(\eta_a)_{\alpha}{}^\gamma, \\ (\Upsilon_\alpha^\perp)_{\bar{\beta}\bar{\gamma}} \Omega^{\bar{\beta}\bar{\gamma}\bar{\delta}} &= \frac{e_i t^i}{\mathcal{K}} v^i (\omega_i)_{\alpha}{}^{\bar{\delta}}. \end{aligned} \quad (3.14)$$

Earlier, in eqs. (2.14) and (2.15), we have split the internal gauge field A_m as $A_m = A_m^{(0)} + A_m^{(1)}$ into a background term $A^{(0)}$ and fluctuation term $A^{(1)}$, which is linear in the gauge matter fields C and D . Comparison of (3.14) with the Ansatz for $A^{(1)}$ shows that Υ^\perp can indeed be absorbed into a re-definition of the matter fields C and D , as claimed earlier. This means, we can set the background gauge field to $w_m^{(T)} + \Upsilon_m^\parallel$ instead of \tilde{w} and write for the gauge field Ansatz

$$A_m = A_m^{(0)} + A_m^{(1)}, \quad (3.15)$$

with

$$A_m^{(0)} = \left(w_{m\alpha\bar{\beta}}^{(T)} + \Upsilon_{m\alpha\bar{\beta}}^\parallel \right) S^{\alpha\bar{\beta}}, \quad (3.16)$$

⁴Note that, although the torsion τ is supposed to be an element of $\mathfrak{su}(3)^\perp$ and, hence, $\tau^\parallel = 0$, in general the parallel component of the con-torsion, κ^\parallel is non-zero. Indeed, from eq. (3.7), we find $\kappa_{\alpha\beta\bar{\gamma}} = -\tau_{\alpha\bar{\gamma}\beta} \neq 0$. Moreover, since the singlet part of κ^\perp vanishes, this component of κ must be part of κ^\parallel .

and $A^{(1)}$ as in eq. (2.15), but with C and D re-interpreted. Here $S^{\alpha\bar{\beta}}$ are the generators of $SU(3)$ in the branching $E_8 \rightarrow SU(3) \otimes E_6$ (see appendix A.3). The background $A_m^{(0)}$ now takes values in $\mathfrak{su}(3)$ and, hence, the low-energy gauge group is E_6 , in line with the perturbative approach. Since we are simply shifting Υ^\perp between background and fluctuations without changing the total gauge field, there should be no problem with this procedure. However, there is one practical difficulty. Given the choice (3.16) for $A^{(0)}$, the background part of the Chern-Simons terms in the definition of H , eq. (2.3), does not cancel by itself. Let us look at the order of the remainder. The perpendicular part κ^\perp of the torsion is linear in the torsion parameters e_i and its contribution to the Chern-Simons term is of $\mathcal{O}(e_i^2)$. The r.h.s. of eq. (2.3) is suppressed by α' , so the resulting contribution to H is of $\mathcal{O}(\alpha' e_i^2)$. Inserting this contribution to H into the Gukov formula (1.1) a non-vanishing contribution of $\mathcal{O}(\alpha' e_i^3)$ can arise from multiplication with dJ (which by itself is of order e_i). This is two powers higher in flux than the terms we keep in our calculation and will, hence, be discarded.

To summarise this section, we have chosen the background gauge fields to be the $\mathfrak{su}(3)$ -valued connection (3.16) so that the resulting four-dimensional gauge group is E_6 . This can be viewed as a standard embedding of \tilde{w} into the gauge field but with the E_6 breaking part, Υ_m^\perp , of the connection being absorbed into a redefinition of the charged fields C and D . This also represents a solution to the Bianchi identity (2.7) at order $\mathcal{O}(\alpha' e_i)$ and therefore it constitutes a consistent background at this order.

3.3 Gravitino mass term at order α'

Having fixed the gauge field Ansatz, we can now proceed and evaluate formula (1.1) for half-flat manifolds at order α' . However, we have to keep in mind that our manifolds are no longer complex but almost complex only. This means that the complex coordinate types of the field strength F cannot be simply obtained by taking holomorphic or anti-holomorphic derivatives of the gauge field in complex coordinates. Rather, we should first re-write the gauge field Ansatz in real coordinates, then differentiate to compute the field strength in real coordinates and only afterwards project to complex coordinate types. In real coordinates, the gauge field Ansatz (3.15), (3.16), (2.15) reads

$$A_m = A_m^{(0)} + A_m^{(1)}, \tag{3.17}$$

with

$$A_m^{(0)} = \left(w_{m\alpha\bar{\beta}}^{(T)} + \Upsilon_{m\alpha\bar{\beta}}^\parallel \right) S^{\alpha\bar{\beta}} \tag{3.18}$$

$$A^{(1)} = \|\Omega\|^{-1/3} (\omega_i)_m{}^n \left(\bar{T}_{n\bar{P}} C^{i\bar{P}} + T_{nP} \bar{C}^{iP} \right) \tag{3.19}$$

$$+ \|\Omega\|^{1/3} \left[(\eta_a)_m{}^n T_{nP} D^{aP} + (\bar{\eta}_{\bar{a}})_m{}^n \bar{T}_{n\bar{P}} \bar{D}^{\bar{a}\bar{P}} \right]. \tag{3.20}$$

Here, we have adopted the convention that the antiholomorphic pieces of the generators T_{nP} , corresponding to the $SU(3) \otimes E_6$ representations $(\bar{\mathbf{3}}, \mathbf{27})$ and $(\mathbf{3}, \mathbf{27})$, vanish, that is

$$T_{\bar{a}P} = \bar{T}_{\alpha\bar{P}} = 0. \tag{3.21}$$

More generally, having to work in real coordinates means that we have to be careful when comparing to the Calabi-Yau formulae in the previous section and convert them into complex coordinates first.

Our first task is to calculate the internal part of the gauge field strength F . To focus our discussion, let us for a moment assume that the background gauge field $A^{(0)}$ equals the $SU(3)$ connection $w^{(T)}$, that is, let us discard the Υ^{\parallel} piece in eq. (3.18) for now. Then, the computation of the field strength is very similar to the computation we have already described for Calabi-Yau manifolds. In particular, we can use the covariant derivative with torsion, $\nabla^{(T)}$, associated to $w^{(T)}$, to re-write the exterior derivative as

$$(dA)_{mn} = \nabla_m^{(T)} A_n - \nabla_n^{(T)} A_m + 2\tau_{mn}{}^p A_p. \quad (3.22)$$

This means, apart from the torsion term on the right hand side which we have to subtract, the formula for the field strength should be the same as in the Calabi-Yau case but with the ordinary covariant derivatives replaced by torsion covariant derivatives. Then, finally, we also have to take into account the effect of a non-vanishing Υ^{\parallel} in (3.18). As in the Calabi-Yau case, it will be useful to organise terms according to their power in the matter fields C and D . Terms in F related to Υ^{\parallel} will be either pure background terms or linear in the matter fields. The pure background terms are not particularly interesting for us. In the Gukov formula (1.1) they lead to background terms which cancel up to higher order terms and to terms linear in the matter fields which should be zero as a consequence of gauge invariance. The linear, Υ^{\parallel} related terms in F , on the other hand, only result from the commutator term in the definition of the field strength and, hence, do not involve derivatives. These terms, as in fact the whole commutator in the expression for the field strength, can be easily computed without the detour to real indices. Hence, for now, we will only write the general expressions for these commutator terms in order not to overload the equations. This understood, we find for the (internal) field strength in real indices

$$\begin{aligned} F_{mn} = & F_{mn}^{(0)} + \|\Omega\|^{-1/3} \nabla_m^{(T)} (\omega_i)_n{}^q T_{qP} \bar{C}^{iP} + \|\Omega\|^{-1/3} \nabla_m^{(T)} (\omega_i)_n{}^q \bar{T}_{q\bar{P}} C^{i\bar{P}} - (m \leftrightarrow n) \\ & + 2\|\Omega\|^{-1/3} \tau_{mn}{}^r (\omega_i)_r{}^q T_{qP} \bar{C}^{iP} + 2\|\Omega\|^{-1/3} \tau_{mn}{}^r (\omega_i)_r{}^q \bar{T}_{q\bar{P}} C^{i\bar{P}} \\ & + \|\Omega\|^{1/3} \nabla_m^{(T)} (\eta_a)_n{}^q T_{qP} D^{aP} + \|\Omega\|^{1/3} \nabla_m^{(T)} (\bar{\eta}_{\bar{a}})_n{}^q \bar{T}_{q\bar{P}} \bar{D}^{\bar{a}\bar{P}} - (m \leftrightarrow n) \\ & + 2\|\Omega\|^{1/3} \tau_{mn}{}^r (\eta_a)_r{}^q T_{qP} D^{aP} + 2\|\Omega\|^{1/3} \tau_{mn}{}^r (\bar{\eta}_{\bar{a}})_r{}^q \bar{T}_{q\bar{P}} \bar{D}^{\bar{a}\bar{P}} \\ & + [A_m^{(1)}, A_n^{(1)}] + 2[\Upsilon_m^{\parallel}, A_n^{(1)}], \end{aligned} \quad (3.23)$$

where $F^{(0)}$ denotes the background field strength computed from $A^{(0)}$ in eq. (3.18), and $A^{(1)}$ and Υ^{\parallel} are explicitly given in eqs. (3.20) and (3.13).

Having derived this result for the field strength one could follow the "traditional" route and derive the four-dimensional scalar potential by computing $\text{tr}F^2$ (as well as H^2) and integrate over the internal space. We will indeed compute a few selected terms in the scalar potential in this way later, in order to check our results. However, given the complexity of eq. (3.23) there is no doubt that the full calculation is rather tedious and that reading off the correct definitions of superfields and the superpotential from the result is likely to be difficult. For example, integrating over the internal space in the presence of

an arbitrary number of (2, 1) and (1, 1) forms will lead to integrals which are non-standard even in the Calabi-Yau case. Further, the background curvature $F_{\text{mn}}^{(0)}$ enters the calculation explicitly. Although, the Ricci tensors for manifolds with SU(3) structure in general and half-flat manifolds in particular have been computed in refs. [38, 39], the results are fairly complicated. At any rate, we would need those results for the somewhat unusual connection $w^{(T)} + \Upsilon^{\parallel}$ which are not readily available. To circumvent these obstacles we would like to base our calculation on the Gukov-formula (1.1), which provides direct information about the gravitino mass $m_{3/2} = e^{K/2}W$. As we will see, with a bit more work, the so-obtained result for $m_{3/2}$ can be disentangled and provides information about the Kähler potential and superpotential.

As in the Calabi-Yau case, the superpotential at order α' arises entirely from the (0, 3) part of the Chern-Simons combination in (2.3). Therefore, we only need to know the (0, 2) component, $F_{(0,2)}$, of the field strength which can be derived by projecting the result (3.23) onto the (0, 2) subspace. Note that all derivatives in eq. (3.23) are torsion covariant derivatives which commute with the almost complex structure J . Therefore, converting to complex indices is as straightforward as for normal complex manifolds. The second observation is that due to eq. (3.21), many of the terms in (3.23) vanish, when written in complex indices. With these facts in mind we find

$$\begin{aligned}
 F_{\bar{\alpha}\bar{\beta}} = & 2|\Omega|^{-1/3}\nabla_{[\bar{\alpha}}^{(T)}(\omega_i)_{\bar{\beta}]}^{\bar{\gamma}}\bar{T}_{\bar{\gamma}\bar{P}}C^{i\bar{P}} + 2|\Omega|^{1/3}\nabla_{[\bar{\alpha}}^{(T)}(\eta_a)_{\bar{\beta}]}^{\bar{\gamma}}T_{\bar{\gamma}P}D^{aP} \\
 & + 2|\Omega|^{-1/3}\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}(\omega_i)_{\bar{\gamma}}^{\delta}T_{\delta P}\bar{C}^{iP} + 2|\Omega|^{1/3}\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}(\bar{\eta}_a)_{\bar{\gamma}}^{\delta}\bar{T}_{\delta\bar{P}}\bar{D}^{a\bar{P}} \\
 & + 2|\Omega|^{-1/3}\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}(\omega_i)_{\bar{\gamma}}^{\delta}\bar{T}_{\delta\bar{P}}C^{i\bar{P}} + 2|\Omega|^{1/3}\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}(\eta_a)_{\bar{\gamma}}^{\delta}T_{\delta P}D^{aP} \\
 & + [A_{\bar{\alpha}}^1, A_{\bar{\beta}}^1] + 2[\Upsilon_{\bar{\alpha}}^{\parallel}, A_{\bar{\beta}}^1].
 \end{aligned} \tag{3.24}$$

Let us compare this with the analogous formula (2.23) on a Calabi-Yau manifold. As we can see the Calabi-Yau result corresponds to the first commutator term in the last line only, while all other terms are new. Specifically, the first line vanishes in the Calabi-Yau case since the covariant derivatives can be reduced to exterior derivatives which act on closed forms. The second and third line obviously vanish for vanishing torsion τ . From eq. (3.13) the tensor Υ^{\parallel} vanishes on a Calabi-Yau space (in the absence of H -flux) and, hence, the last term also disappears in this case. Another important remark about the above result concerns the origin of the second line. For complex manifolds this line would vanish identically as, in this case, the (0, 2) component of the field strength can be constructed from the (0, 1) component of the gauge field alone. Then, $F_{(0,2)}$ would depend on C and D only but not on their complex conjugates. This shows that our detour to real indices has been important and, without it, we would have missed the second line of the above result.⁵

For completeness, we also present the expression for the (1, 1) component, $F_{(1,1)}$, of the field strength, although this result will not be needed in the remainder of the section. Note that from eq. (3.5) the first two indices of the intrinsic torsion are both holomorphic or anti-holomorphic. Therefore, the second and fourth line in eq. (3.23) do not contribute

⁵This can also be seen formally by observing that the torsion components $\tau_{\alpha\beta}^{\bar{\gamma}}$ are directly related to the lack of integrability of the almost complex structure [19].

to $F_{(1,1)}$ and we are left with

$$\begin{aligned}
 F_{\alpha\bar{\beta}} &= F_{\alpha\bar{\beta}}^0 + \|\Omega\|^{-1/3} \nabla_{\alpha}^{(T)}(\omega_i)_{\bar{\beta}}^{\bar{\gamma}} \bar{T}_{\bar{\gamma}\bar{P}} C^{i\bar{P}} + \|\Omega\|^{1/3} \nabla_{\alpha}^{(T)}(\eta_a)_{\bar{\beta}}^{\gamma} T_{\gamma P} D^{aP} \\
 &\quad - \|\Omega\|^{-1/3} \nabla_{\bar{\beta}}^{(T)}(\omega_i)_{\alpha}^{\gamma} T_{\gamma P} \bar{C}^{iP} - \|\Omega\|^{1/3} \nabla_{\bar{\beta}}^{(T)}(\bar{\eta}_{\bar{a}})_{\alpha}^{\bar{\gamma}} \bar{T}_{\bar{\gamma}\bar{P}} \bar{D}^{\bar{a}\bar{P}} \\
 &\quad + [A_{\alpha}^1, A_{\bar{\beta}}^1] + [A_{\alpha}^1, \bar{K}_{\bar{\beta}}^{\parallel}] + [\Upsilon_{\alpha}^{\parallel}, A_{\bar{\beta}}^1] .
 \end{aligned} \tag{3.25}$$

Terms proportional to E_6 generators in this expression can only arise from the first commutator term in the last line, just as for Calabi-Yau manifolds. Therefore, by virtue of eq. (2.28), the D -terms will be unchanged from the Calabi-Yau case and are given by eq. (2.30).

Let us now compute the $(0,3)$ component of the Chern-Simons term. Cubic terms in the matter fields only arise from $(A^{(1)})^3$ and should, therefore, be unchanged from the Calabi-Yau case. This means the standard cubic terms (2.26) in the superpotential are also present in the half-flat case. Terms which do not contain charged fields cancel up to higher order terms, while linear terms are absent due to gauge invariance. Thus the only new terms we can expect in the superpotential are quadratic terms in the gauge matter fields. They arise from linear matter field terms in $F_{(0,2)}$, that is the first three lines of eq. (3.24) and the last term involving Υ^{\parallel} , multiplied with $A^{(1)}$, as well as from the $\Upsilon^{\parallel} \wedge (A^{(1)})^2$ term contained in A^3 .⁶

Let us denote by $F^{(1)}$ the part of $F_{(0,2)}$ linear in matter fields but excluding the contributions from Υ^{\parallel} for now. The quadratic matter terms in the Yang-Mills Chern-Simons form not related to Υ^{\parallel} can then be written as

$$\begin{aligned}
 \text{tr}(F^{(1)} \wedge A^{(1)})_{\bar{\alpha}\bar{\beta}\bar{\gamma}} &= 6\|\Omega\|^{-2/3} \tau_{[\bar{\alpha}\bar{\beta}}^{\gamma}(\omega_i)_{\bar{\gamma}]}^{\bar{\delta}}(\omega_j)_{\gamma\bar{\delta}} C^{i\bar{P}} \bar{C}^j_{\bar{P}} \\
 &\quad + 6\|\Omega\|^{2/3} \tau_{[\bar{\alpha}\bar{\beta}}^{\gamma}(\eta_a)_{\bar{\gamma}]}^{\bar{\delta}}(\bar{\eta}_{\bar{b}})_{\gamma\bar{\delta}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} \\
 &\quad + 6 \left[\nabla_{[\bar{\alpha}}^{(T)}(\eta_a)_{\bar{\beta}}^{\gamma}(\omega_i)_{\bar{\gamma}]}^{\bar{\delta}} + \tau_{[\bar{\alpha}\bar{\beta}}^{\bar{\delta}}(\omega_i)_{\bar{\gamma}]}^{\gamma}(\eta_a)_{\bar{\delta}}^{\gamma} \right. \\
 &\quad \left. + \nabla_{[\bar{\alpha}}^{(T)}(\omega_i)_{\bar{\beta}]}^{\gamma}(\eta_a)_{\bar{\gamma}]}^{\bar{\delta}} + \tau_{[\bar{\alpha}\bar{\beta}}^{\bar{\delta}}(\eta_a)_{\bar{\gamma}]}^{\gamma}(\omega_i)_{\bar{\delta}}^{\gamma} \right] C^i_{\bar{P}} D^{aP} ,
 \end{aligned} \tag{3.26}$$

where we have used the trace formula (A.29).

To obtain the superpotential we have to integrate the contraction of this formula with $\Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}}$. In the second line, we can integrate by parts to move the derivative to ω_i and we obtain

$$\begin{aligned}
 \int \text{tr}(F^{(1)} \wedge A^{(1)}) \wedge \Omega &= i \int \tau_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}} \left[(\omega_i)_{\bar{\gamma}\gamma}(\eta_a)_{\bar{\delta}}^{\gamma} - (\omega_i)_{\bar{\delta}\gamma}(\eta_a)_{\bar{\gamma}}^{\gamma} \right] \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} C^i_{\bar{P}} D^{aP} \\
 &\quad + 2i \int \left[\nabla_{\bar{\alpha}}^{(T)}(\omega_i)_{\bar{\beta}]}^{\gamma}(\eta_a)_{\bar{\gamma}}^{\bar{\delta}} + \tau_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}}(\eta_a)_{\bar{\gamma}}^{\gamma}(\omega_i)_{\bar{\delta}}^{\gamma} \right] \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} C^i_{\bar{P}} D^{aP} \\
 &\quad + i\|\Omega\|^{-2/3} \int \tau_{\bar{\alpha}\bar{\beta}}^{\gamma}(\omega_i)_{\bar{\gamma}}^{\bar{\delta}}(\omega_j)_{\gamma\bar{\delta}} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} C^{i\bar{P}} \bar{C}^j_{\bar{P}} \\
 &\quad + i\|\Omega\|^{2/3} \int \tau_{\bar{\alpha}\bar{\beta}}^{\gamma}(\eta_a)_{\bar{\gamma}}^{\bar{\delta}}(\bar{\eta}_{\bar{b}})_{\gamma\bar{\delta}} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} .
 \end{aligned} \tag{3.27}$$

⁶Note that quadratic terms in the charged fields which involve the $SU(3)$ spin connection should vanish by the same arguments as in the Calabi-Yau case.

This formula looks complicated, but there are a number of simplifications. Recall from (3.5) that $\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}}$ is a primitive (1, 2) form and, therefore, the combination $\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}}\Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ is symmetric in the indices $(\bar{\delta}, \bar{\gamma})$. On the other hand, the bracket in the first line is explicitly anti-symmetric in these indices. Hence, the first line vanishes. Further, in the second line, the indices $[\bar{\alpha}, \bar{\beta}]$ are antisymmetrised so that the covariant derivative can be converted into an exterior derivative. Explicitly, we have

$$2\nabla_{[\bar{\alpha}}^{(T)}\omega_{\bar{\beta}]\gamma} = (d\omega)_{\bar{\alpha}\bar{\beta}\gamma} - 2\tau_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}}\omega_{\bar{\delta}\gamma}, \tag{3.28}$$

and, therefore, the torsion drops out from the second line and only $d\omega_i$ appears. This can be replaced using the half-flat mirror relations (3.3) which reduces the second line to the integral

$$ie_i \int (\beta^0)_{\bar{\alpha}\bar{\beta}\gamma} (\eta_a)_{\bar{\gamma}} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} = 2e_i K_a. \tag{3.29}$$

Here K_a denotes the derivative of the complex structure Kähler potential $K_a = \frac{\partial K(z)}{\partial z^a}$. In order to carry out the integral, we have used the definition (A.15) of η_a and the Kodaira formula (A.8) as well as the standard choice $\mathcal{Z}^0 = 1$.

Now we are left with having to evaluate the last two lines in eq. (3.27). First note from (3.5) that the contraction of the torsion $\tau_{\bar{\alpha}\bar{\beta}}^{\gamma}$ with $\Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ can be written as

$$\tau_{\bar{\alpha}\bar{\beta}}^{\gamma}\Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \frac{i}{4\mathcal{K}} e_i g^{ij} (\omega_j)^{\gamma\bar{\gamma}}. \tag{3.30}$$

Then, the last two lines in eq. (3.27) lead to the integrals

$$\sigma_{lij} = \frac{i}{4\mathcal{K}} \int (\omega_l)^{\gamma\bar{\gamma}} (\omega_i)_{\bar{\gamma}}^{\bar{\delta}} (\omega_j)_{\gamma\bar{\delta}}, \tag{3.31}$$

and

$$\tilde{\sigma}_{jab} = \frac{i}{4\mathcal{K}} \int (\omega_j)^{\gamma\bar{\gamma}} (\eta_a)_{\bar{\gamma}}^{\delta} (\bar{\eta}_b)_{\gamma\delta}. \tag{3.32}$$

These integrals are non-standard on a Calabi-Yau manifold and presumably difficult to compute. However, we will not need their general values but merely their contractions with the Kähler moduli v^i . Using (A.3), (A.6) and (A.16) these contractions can be explicitly computed and we find

$$v^l \sigma_{lij} = g_{ij}, \quad v^j \tilde{\sigma}_{jab} = \frac{1}{4} g_{ab}, \tag{3.33}$$

where g_{ij} and g_{ab} are the Kähler and complex structure moduli space metrics.

Combining these results we can finally write eq. (3.27) as

$$\int (F^{(1)} \wedge A^{(1)}) \wedge \Omega = \alpha' \left[ie_k g^{kl} \left(\|\Omega\|^{-2/3} \sigma_{lij} C^{i\bar{P}} \bar{C}^j_{\bar{P}} + \|\Omega\|^{2/3} \tilde{\sigma}_{lab} D^{aP} \bar{D}^b_{\bar{P}} \right) + 2e_i K_a C^{i\bar{P}} D^a_{\bar{P}} \right]. \tag{3.34}$$

To obtain all quadratic matter field terms in the gravitino mass we still have to add the terms related to Υ^{\parallel} . They arise from terms of the form $\Upsilon^{\parallel} \wedge (A^{(1)})^2$ which are contained in both the $F \wedge A$ and A^3 terms of the Yang-Mills Chern-Simons form. Taking all the

factors into account we get the following contribution to the gravitino mass from terms proportional to Υ^\parallel

$$\begin{aligned} \int \omega_{\text{YM}} \wedge \Omega \Big|_{\Upsilon} &= -2i\alpha' \int \Upsilon^\parallel_{\bar{\alpha}\alpha\bar{\delta}}(\omega_i)_{\bar{\beta}}^{\bar{\delta}}(\eta_a)_{\bar{\gamma}}^{\alpha} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} C_P^i D^{aP} \\ &= i\alpha' e_j t^j \int (\beta^0)_{\bar{\alpha}\alpha\bar{\delta}}(\omega_i)_{\bar{\beta}}^{\bar{\delta}}(\eta_a)_{\bar{\gamma}}^{\alpha} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} C_P^i D^{aP}, \end{aligned} \quad (3.35)$$

where we have used eq. (3.13). This integral is similar to the ones discussed above and can, in fact, be written in terms of $\tilde{\sigma}_{i\bar{a}\bar{b}}$ defined in (3.32). However, we will not need its explicit form and simply introduce the symbol $\Sigma_{i\bar{a}}$

$$\Sigma_{i\bar{a}} = i \int (\beta^0)_{\bar{\alpha}\alpha\bar{\delta}}(\omega_i)_{\bar{\beta}}^{\bar{\delta}}(\eta_a)_{\bar{\gamma}}^{\alpha} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}}, \quad (3.36)$$

such that

$$\int \omega_{\text{YM}} \wedge \Omega \Big|_{\Upsilon} = \alpha' e_j t^j \Sigma_{i\bar{a}} C_P^i D^{aP}. \quad (3.37)$$

To summarise this section let us write the final formula for the gravitino mass at order α' which is given by the sum of eqs. (2.26), (3.4), (3.34) and (3.37) and reads

$$\begin{aligned} m_{3/2} \equiv e^{K/2} W &= e^{K_0/2} \left\{ e_i t^i - \frac{1}{3} \alpha' \left[\bar{j}_{\bar{P}\bar{R}\bar{S}} \mathcal{K}_{ijk} C^{i\bar{P}} C^{j\bar{R}} C^{k\bar{S}} + j_{\text{PRS}} \tilde{\mathcal{K}}_{abc} D^{aP} D^{bR} D^{cS} \right] \right. \\ &\quad \left. + \alpha' \left[i e_k g^{kl} \left(\|\Omega\|^{-2/3} \sigma_{lij} C^{i\bar{P}} \bar{C}^j_{\bar{P}} + \|\Omega\|^{2/3} \tilde{\sigma}_{l\bar{a}\bar{b}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} \right) \right. \right. \\ &\quad \left. \left. + 2e_i K_a C^{i\bar{P}} D^a_{\bar{P}} \right] + \alpha' e_i t^i \Sigma_{j\bar{a}} C_P^j D^{aP} \right\}. \end{aligned} \quad (3.38)$$

Here, K_0 stands for the Kähler potential at zeroth order in α' . This is the one of the main results of the paper. In the following sections we will analyse its interpretation and implications for the four-dimensional effective theory.

3.4 Four-dimensional effective theory

Eq. (3.38) provides us with the with the supergravity G -function, $G = K + \ln|W|^2 = \ln|m_{3/2}|^2$. This, together with the gauge kinetic function which has already been computed in ref. [22], completely determines the four-dimensional supergravity Lagrangian. It seems, we should, therefore, be able to find the complete low-energy theory from the results so far. However, eq. (3.38) as stands is still expressed in terms of the 10-dimensional fields and it first needs to be re-written in terms of the correct four-dimensional superfields. In other words, we need to know the definition of the four-dimensional superfields in terms of the underlying 10-dimensional fields. It is not obvious that this information can be extracted from the above results. To analyse the situation it is useful to compare eq. (3.38) with a general expression for the gravitino mass, expanded to linear order in α' . Let us denote by ϕ_0 , K_0 and W_0 the moduli fields, the Kähler potential and the superpotential at zeroth

order in α' and by ϕ , K and W their counterparts at order first order in α' . We can write

$$\begin{aligned}\phi &= \phi_0 + \alpha' \delta\phi, \\ K(\phi) &= K_0(\phi_0) + \alpha' \delta K, \\ W(\phi) &= W_0(\phi_0) + \alpha' (\delta W + \partial_\phi W \delta\phi),\end{aligned}\tag{3.39}$$

where $\delta\phi$ is a correction to the definition of the moduli fields which is expected [28] at order α' . Further δK and δW are the changes of the Kähler potential and superpotential⁷ at order α' . The gravitino mass can then be expanded as⁸

$$m_{3/2} = e^{K/2} W = e^{K_0(\phi_0)/2} [W_0(\phi_0) + \alpha' (\delta W + W \delta K + \partial_\phi W \delta\phi)] .\tag{3.40}$$

Let us now compare this general expression to our explicit result for the gravitino mass (3.38). Clearly, the first term in eq. (3.38) corresponds to the superpotential at zeroth order in α' , that is, to the term $W_0(\phi_0)$ in our general notation. The rest of the first line is the well-known cubic superpotential for the matter fields which arises at order α' and it should be part of δW . All other terms in eq. (3.38) are of order α' and non-holomorphic and, hence, must correspond to the last two terms in eq. (3.40), that is, they must be due to α' corrections to the definition of the moduli fields or to the Kähler potential. Given that we have a Kähler moduli superpotential at zeroth order in α' we indeed need correction terms which convert this superpotential into a function of the proper order α' superfields. What we have to decide is which of the terms in the second and third line of (3.38) are absorbed into a re-definition of the Kähler moduli t^i . It turns out the correct choice is to absorb all terms in the second line of (3.38) into t^i while interpreting the term in the third line as a correction to the Kähler potential. This is suggested by the fact that corrections to K should appear multiplied with W (as in eq. (3.40)) which is only the case for the last term in (3.38). Also, we know from the standard Calabi-Yau case [28] that the first two terms in the second line should definitely be part of the re-definition of t^i , so the only non-trivial question is really about the last two terms in (3.38). We will check towards the end of this section that our choice for these remaining two terms is indeed correct.

Let us now formalise the previous discussion. We write the relation between the zeroth order Kähler moduli t^i and their order α' counterparts T^i as

$$T^i = t^i + \alpha' Y^i .\tag{3.41}$$

where the correction terms Y^i are explicitly given by

$$Y^i = i g^{ij} (|\Omega|^{-2/3} \sigma_{jkl} C^{k\bar{P}} \bar{C}^l_{\bar{P}} + |\Omega|^{2/3} \tilde{\sigma}_{j\bar{a}\bar{b}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}}) + 2K_a C^{i\bar{P}} D^a_{\bar{P}} .\tag{3.42}$$

From eq. (3.38), the superpotential is then given by

$$W = e_i T^i - \frac{\alpha'}{3} \tilde{j}_{\bar{P}\bar{R}\bar{S}} \mathcal{K}_{ijk} C^{i\bar{P}} C^{j\bar{R}} C^{k\bar{S}} - \frac{\alpha'}{3} j_{\text{PRS}} \tilde{\mathcal{K}}_{abc} D^{aP} D^{\text{bR}} D^{\text{cS}} .\tag{3.43}$$

⁷It is convenient to separate out the change of the superpotential due to the re-definition of the moduli fields explicitly while writing the change in the Kähler potential as a single term.

⁸Since we are working to first order in α' , we do not need to distinguish between corrected and uncorrected quantities in the order α' part of this expression.

Note that the torsion part of the superpotential has absorbed the terms in the second line of (3.38) and, as a result, is now expressed in terms of the corrected superfields, T^i , as is should.

The only remaining term is the last one in (3.38). It gives rise to a Kähler potential correction so that the total Kähler potential, $K = K_0(\phi_0) + \delta K$, can be written as

$$K = K_0(s, \bar{s}, t, \bar{t}, z, \bar{z}) + \alpha' \left[\Sigma_{ia} C_P^i D^{aP} + \text{c.c.} \right], \quad (3.44)$$

where the moduli Kähler potential K_0 is given in eq. (2.9). The Kähler moduli part, K_K , of K_0 still needs to be expressed in terms of the corrected moduli fields T^i , so we write

$$K_K(t, \bar{t}) = K_K(T - \alpha' Y, \bar{T} - \alpha' \bar{Y}) = K_K(T, \bar{T}) - \alpha' K_i Y^i - \alpha' \bar{K}_i \bar{Y}^i + \mathcal{O}(\alpha'^2), \quad (3.45)$$

where K_i is the derivative of the Kähler potential with respect to t^i . Using the Calabi-Yau identity

$$K_i g^{ij} = 2i v^j, \quad (3.46)$$

together with eqs. (3.33) we find

$$K_i Y^i = -2 \|\Omega\|^{-2/3} g_{kl} C^{k\bar{P}} \bar{C}^l_{\bar{P}} - \frac{1}{2} \|\Omega\|^{2/3} g_{a\bar{b}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} + 2 K_i K_a C^{i\bar{P}} D^a_{\bar{P}}. \quad (3.47)$$

The Kähler potential (3.44) can then be written as

$$K = K_0(s, \bar{s}, T, \bar{T}, z, \bar{z}) + \alpha' \left[4 \|\Omega\|^{-2/3} g_{ij} C^{i\bar{P}} \bar{C}^j_{\bar{P}} + \|\Omega\|^{2/3} g_{a\bar{b}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} + ((\Sigma_{ia} - 2 K_i K_a) C^i_{\bar{P}} D^{aP} + \text{c.c.}) \right]. \quad (3.48)$$

Having corrected the Kähler moduli at order α' it seems likely the same has to be done to the complex structure moduli, so we write the α' corrected complex structure moduli Z^a as

$$Z^a = z^a + \alpha' Y^a. \quad (3.49)$$

From the result (3.38) we have no direct information about the corrections Y^a since the zeroth order superpotential is independent of the complex structure moduli. One guess might be that the additional δK term (the term proportional to Σ_{ia} in eq. (3.48)) we have found is responsible for the re-definition of the complex structure moduli. This would imply that

$$Y^a = \Sigma_{ia}{}^b C^i_{\bar{P}} D^{aP} \quad (3.50)$$

for some tensor $\Sigma_{ia}{}^b$ with the property $K_b \Sigma_{ia}{}^b = \Sigma_{ia}$. Later we will introduce H -flux which provides us with a zeroth order superpotential for the complex structure moduli and explicit information about the redefinition of z^a . We will then confirm the above expression for Y^a . Accepting our guess for now we can write the Kähler potential as

$$K = K_0(s, \bar{s}, T, \bar{T}, Z, \bar{Z}) + \alpha' \left[4 \|\Omega\|^{-2/3} g_{ij} C^{i\bar{P}} \bar{C}^j_{\bar{P}} + \|\Omega\|^{2/3} g_{a\bar{b}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} - 2 (K_i K_a C^i_{\bar{P}} D^{aP} + \text{c.c.}) \right]. \quad (3.51)$$

Eqs. (3.51) and (3.43) represent our final result for the order α' Kähler potential and superpotential from standard embedding compactifications of the heterotic string on half-flat mirror manifolds. The matter field part is E_6 invariant and identical to the one found for Calabi-Yau compactifications with standard embedding. The only difference to the Calabi-Yau case is the zeroth order torsion superpotential for the Kähler moduli which is simply added to the standard cubic superpotential for the matter fields. Although the CD terms in the Kähler potential (3.51) (and the field re-definitions (3.42)) look unconventional they are independent of the torsion parameters e_i and should, therefore, be already present in the Calabi-Yau case. To our knowledge they have not been explicitly computed before, although their possible existence has been anticipated in ref. [33]. These terms do not contribute to the matter field kinetic terms (although they do contribute to mixed matter field/moduli kinetic terms) and, in the Calabi-Yau case, they affect the scalar potential only at higher order in α' . It is not surprising, therefore, that these terms are usually omitted. A curious feature of our result is that the order α' re-definition of Kähler and complex structure moduli is quite different, see eqs. (3.42) and (3.50). In particular, $C\bar{C}$ and $D\bar{D}$ terms appear for the Kähler moduli only. This means that the standard kinetic terms for both the (1, 1) and (2, 1) matter fields are linked to the re-definition of the (1, 1) moduli.

Should we have expected new terms in the matter field sector compared to the Calabi-Yau case? Given that our set-up leads to an E_6 invariant low-energy theory, the only additional terms allowed from gauge invariance are $C^i{}_P D^{aP}$ terms in the superpotential. We know that such terms are definitely absent in the Calabi-Yau case and this can be understood from the fact that we have required that the matter fields C and D in eq. (2.15) be massless. By turning on fluxes we may expect that some of these fields become massive, but the above calculation shows that a supersymmetric mass term is not generated.⁹ We interpret this as an indication that 2-index couplings (fluxes), λ_{ia} , are needed for these terms to appear in the superpotential. We will in fact see that for the generalised half-flat manifolds discussed in section 4, for which torsion parameters have one Kähler and one complex structure index, CD superpotential terms indeed arise.

One obvious simple check of our results is to compare the D-term, as obtained from the Gukov-type formula (2.28), with the four-dimensional supergravity expression (2.27) after inserting the above results for Kähler potential and superpotential. We recall that the Gukov-type formula predicts the D-terms for half-flat mirror manifolds should be unchanged from the Calabi-Yau case. This is indeed what one finds when inserting (3.51) and (3.43) into the supergravity formula (2.27).

3.5 Including H-flux

An obvious extension of our set-up is to include NS-NS flux. This will generate a zeroth order superpotential for the complex structure moduli, in addition to the Kähler moduli superpotential from torsion already present. As indicated before, this can provide us with

⁹It is not hard to see that at the level of the $N = 1$ potential, terms of the type $(e_i T^i) g_{jk} C_P^j \bar{C}^{kP}$ and similar ones for the D -fields are in fact generated at first order in α' making these fields indeed massive. Note that this is only possible at this order in α' due to the appearance of the zeroth order superpotential (3.4).

additional information about the complex structure moduli and consistency checks of our results.

For simplicity, we start with NS-NS flux of the form $H_{\text{flux}} = p_a \beta^a$, with flux parameters p_a . This leads to a zeroth order superpotential contribution

$$W_{0,\text{flux}} = \int \Omega \wedge H = p_a z^a . \quad (3.52)$$

which, in analogy with the torsion superpotential, is linear in the moduli.

Does the NS-NS flux lead to any corrections at first order in α' ? We have to remember that, via eq. (2.6), H appears in the connection which enters the Bianchi identity. The resulting change in the gravitino mass can be computed directly from eq. (3.35) with Υ replaced by H_{flux} and making use of formulae (A.9). Together with the zeroth order term (3.52) this leads to the following additional terms in the gravitino mass due to flux

$$e^{K/2} W \Big|_{\text{flux}} = e^{K_0/2} \left[p_a z^a + \alpha' (p_b + p_c z^c K_b) \Sigma_{\text{ia}}{}^b C_P^i D^{\text{aP}} \right] , \quad (3.53)$$

where we have defined the quantity

$$\Sigma_{\text{ia}}{}^b = - \frac{i}{\int \Omega \wedge \bar{\Omega}} g^{\bar{a}b} \int (\bar{\chi}_{\bar{a}})_{\bar{\alpha}\alpha\bar{\delta}} (\omega_i)_{\bar{\beta}}{}^{\bar{\delta}} (\eta_a)_{\bar{\gamma}}{}^{\alpha} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} . \quad (3.54)$$

Let us now analyse this result by comparing it to the general expression (3.40) for the gravitino mass, as we did before. The order α' terms in eq. (3.53) are non-holomorphic and, hence, they should correspond to either corrections to the Kähler potential or re-definitions of moduli fields. The last term in eq. (3.53) is proportional to the flux superpotential and this suggests it should be viewed as a correction to the Kähler potential. This interpretation is, in fact, required for consistency, given that we have declared the last term in (3.38) to be a Kähler potential correction as well. Indeed, in the presence of flux, we need another term to combine with the last one in eq. (3.38) to produce the total torsion and flux superpotential as a pre-factor. One can check, using relation (A.9), that

$$K_b \Sigma_{\text{ia}}{}^b = \Sigma_{\text{ia}} , \quad (3.55)$$

with Σ_{ia} defined in eq. (3.36) which provides a confirmation of this interpretation. The second term in eq. (3.53), on the other hand, coincides with the correction (3.50) to the definition of the complex structure moduli which we have anticipated earlier. Hence, this term combines with the flux superpotential and changes the moduli z^a into their α' corrected counterparts Z^a .

To summarise, we have confirmed our earlier result (3.51) for the Kähler potential and the superpotential is given by (3.43) plus the addition flux contribution

$$W_{\text{flux}} = p_a Z^a . \quad (3.56)$$

3.6 Consistency with compactification results

The identification of low-energy data from the gravitino mass (3.38), (3.53) in the last subsection has, in parts, relied on a suggestive interpretation rather than strict conclusion.

It would, therefore, be desirable to have an independent and meaningful check through a direct compactification of the 10-dimensional theory. We have argued before that this is a difficult task, firstly due to the large number of terms in the potential and secondly due to the presence of certain integrals which are not standard on Calabi-Yau manifolds. We have already encountered such integrals in the calculation of the gravitino mass, although we have managed to proceed without knowing their explicit form. One way to simplify the calculation is to consider an associated Calabi-Yau manifold with only a single Kähler modulus. Then the forms ω_i can be replaced by the almost complex structure J and the integrals involved become significantly simpler. The number of terms is also reduced, although it is still considerable. In addition to assuming $h^{1,1} = 1$, we will, therefore, focus on a specific class of terms, namely scalar potential terms of the form $DD\bar{C}$ and $CC\bar{D}$ together with their complex conjugates.

Let us explain how these terms appear when compactifying the 10-dimensional action. First of all, cubic terms in the matter fields appear from F^2 , taking terms linear in the charged fields in one F (either from the explicit terms in (3.23) or from the last commutator) and terms quadratic in the matter fields in the second F (from the first commutator in (3.23)). Another source for cubic matter field terms is H^2 with one H taken to be the zeroth order part (3.8) from torsion and the other the Chern-Simons form (in fact only the term A^3 can contribute). For the case of one Kähler modulus, it is not hard to check that all of the derivative terms from (3.23) drop out — either directly or after integration by parts — and only the linear terms with no derivative contribute. After a long but straightforward calculation one obtains

$$V = \dots + \alpha' \frac{e^{-2\phi_4}}{\mathcal{K}} \left[\frac{1}{\mathcal{K} \|\Omega\|^{8/3}} \left(\frac{3i}{2} \frac{eb}{v} + \frac{e}{2} \right) K_{\bar{c}} g^{\bar{c}c} \tilde{\mathcal{K}}_{abcj} \text{PR}S D^{aP} D^{bR} \bar{C}^S \right. \\ \left. + \frac{v^2}{3\mathcal{K} \|\Omega\|^{4/3}} \left(\frac{3i}{2} \frac{eb}{v} + \frac{e}{2} \right) K_{\bar{a}j} \bar{P} \bar{R} \bar{S} C^{\bar{P}} C^{\bar{R}} \bar{D}^{\bar{a}\bar{S}} \right] + \text{c.c.} , \quad (3.57)$$

where the dots stand for other type of terms we have not computed here.

On the supergravity side, starting from the Kähler potential (3.51) and superpotential (3.43), we can compute the above terms for general $h^{1,1}$ and only at the very end take the limit of one Kähler modulus. We note that the $|W|^2$ term in the supergravity potential cannot produce the type of terms considered here. Therefore we only have to consider the F -part of the supergravity potential. To get a contribution proportional to C^2 or D^2 to an F-term we need to consider the derivatives $\partial_C W$ or $\partial_D W$, respectively. These derivatives are already at order α' and, hence, need to be multiplied by an order zero piece. It is clear that the CC and DD terms in the matter field Kähler potential cannot lead to cubic terms mixing C and D . Therefore the term CD in the Kähler potential is crucial in order to reproduce the terms in (3.57). To see precisely which terms do contribute one has to compute the Kähler metric, including the complex structure moduli, from (3.51) and invert this metric at order α' . Schematically, one then finds the following relevant terms

$$V \sim D_C W \left(g^{\bar{C}C} D_{\bar{C}} \bar{W} + g^{C\bar{z}} D_{\bar{z}} \bar{W} + g^{C\bar{T}} D_{\bar{T}} \bar{W} \right) , \quad (3.58)$$

plus similar terms with C and D exchanged. From the first term in the bracket one keeps the derivative on the Kähler potential times W_0 , from the second term the derivative on the complex structure Kähler potential times W_0 while the full $D_T \bar{W}_0$ contributes from the last term. Computing explicitly all terms and taking the limit of one Kähler modulus, it is not hard to see that the result indeed reproduces (3.57). This constitutes a powerful check of our results. In particular, it confirms that the CD terms should indeed be present in the Kähler potential (3.51) and that the identifications of various terms in (3.38) was correct.

4. Generalised half-flat manifolds

In the final part of this paper let us discuss an extension of the results obtained in the previous section to more general manifolds with $SU(3)$ structure which we refer to as generalised half-flat manifolds. These manifolds were proposed in ref. [29, 30]. Working out the effective theories associated to these manifolds is not conceptually new, but rather a straightforward generalisation of the results obtained in the previous section.

Let us briefly present the setup for these compactifications, relying on the conventions of ref. [23]. We consider a manifold with $SU(3)$ structure and two-forms ω_i and three-forms (α_A, β^B) which obey the following algebra

$$\begin{aligned} d\omega_i &= p_{iA}\beta^A - q_i^A\alpha_A, \\ d\alpha_A &= p_{iA}\tilde{\omega}^i, \quad d\beta^A = q_i^A\tilde{\omega}^i, \\ d\tilde{\omega}^i &= 0, \end{aligned} \tag{4.1}$$

with the constants p_{iA} and q_i^A subject to the constraints

$$p_{iA}q_j^A - p_{jA}q_i^A = 0. \tag{4.2}$$

These relations replace the analogous relations (3.3) for half-flat mirror manifolds. In addition, it is assumed that the link of these manifolds with the related Calabi-Yau manifolds is precisely as for half-flat mirror manifolds. Half-flat mirror manifolds corresponds to the particular choice $p_{i0} = e_i$, $p_{ia} = 0$ and $q_i^A = 0$.

The zeroth order superpotential for generalised half-flat manifolds is more complex than the one for half-flat mirror manifolds, eq. (3.4), and contains mixed Kähler and complex moduli terms as well. It is given by [23]

$$W_0 = p_{iA}t^i\mathcal{Z}^A - q_i^A t^i\mathcal{G}_A \equiv E_i t^i, \tag{4.3}$$

where \mathcal{G}_A are the derivatives of the complex structure prepotential \mathcal{G} . To make the analysis similar to the half-flat case we have introduced the notation

$$E_i(\mathcal{Z}) = p_{iA}\mathcal{Z}^A - q_i^A\mathcal{G}_A. \tag{4.4}$$

With this notation, much of the calculations for half-flat mirror manifolds can be carried over to the present case by replacing the torsion parameters e_i with E_i . In particular, eq. (3.30) still holds with this replacement. Hence, we can directly obtain the result for

the last integrals in eq. (3.27). Working out the second line in eq. (3.27) however, requires some modifications. The result previously given by eq. (3.29) now changes to

$$i \int (d\omega_i)_{\bar{\alpha}\bar{\beta}\bar{\gamma}}(\eta_a)_{\bar{\gamma}} \Omega^{\bar{\alpha}\bar{\beta}\bar{\gamma}} = -2 \int d\omega_i \wedge \chi_a = 2(p_{ia} - q_i^B \mathcal{G}_{Ba}) + 2K_a E_i \quad (4.5)$$

where the last equality follows from straightforward computations using the standard Calabi-Yau relations (A.9). Finally one can show that Υ^{\parallel} in eq. (3.13) can now be written as¹⁰

$$(\Upsilon_{\bar{\alpha}}^{\parallel})_{\alpha\bar{\beta}} = -\frac{1}{2} t^i d\omega_i = -\frac{1}{2} t^i (p_{iA} \beta^A - q_i^A \alpha_A). \quad (4.6)$$

With this, and using again formulae (A.9), the result in eq. (3.37) becomes

$$\int \Omega_{\text{YM}} \wedge \Omega \Big|_{\Upsilon} = \alpha' t^i [(p_{ia} - q_i^A \mathcal{G}_{Aa}) + K_a (p_{ia} \mathcal{Z}^A - q_i^A \mathcal{G}_A)] \Sigma_{\text{jb}}^a C_P^j D^{\text{bP}}. \quad (4.7)$$

Collecting the above contributions, the final formula for the gravitino mass in this more general case then takes the form

$$e^{K/2} W = e^{K_0/2} \left\{ E_i t^i - \frac{1}{3} \alpha' \left[\bar{j}_{\bar{P}\bar{R}\bar{S}} \mathcal{K}_{ijk} C^{i\bar{P}} C^{j\bar{R}} C^{k\bar{S}} + j_{\text{PRS}} \tilde{\mathcal{K}}_{abc} D^{aP} D^{\text{bR}} D^{\text{cS}} \right] \right. \quad (4.8) \\ \left. + \alpha' \left[i E_k g^{\text{kl}} \left(\|\Omega\|^{-2/3} \sigma_{lij} C^{i\bar{P}} \bar{C}^j_{\bar{P}} + \|\Omega\|^{2/3} \tilde{\sigma}_{la\bar{b}} D^{aP} \bar{D}^{\bar{b}}_P \right) \right. \right. \\ \left. \left. + 2E_i K_a C^{i\bar{P}} D^a_{\bar{P}} \right] + \alpha' [(p_{ia} - q_i^A \mathcal{G}_{Aa}) t^i + K_a E_i t^i] \Sigma_{\text{jb}}^a C_P^j D^{\text{bP}} \right. \\ \left. + 2\alpha' (p_{ia} - q_i^B \mathcal{G}_{Ba}) C_P^i D^{\text{aP}} \right\}.$$

Given the expression (4.3) for the torsion superpotential, it follows that the interpretation of most terms above is the same as for the half-flat mirror case (3.38): the first line is part of the superpotential at order α' , the second line should be interpreted as a redefinition of the Kähler moduli t^i like in eqs. (3.41) and (3.42) while the first terms in the last line are analogous to the last terms in eq. (3.53) and correspond to the redefinition of the complex structure moduli (3.50) and the change in the Kähler potential from eq. (3.48). The only difference compared to the cases studied before is the last term above. As it is holomorphic there is no need to absorb it into a redefinition of moduli and it turns to be part of the superpotential. On general grounds, this is actually not surprising given that we now have the couplings p_{iA} and q_i^A which allow holomorphic (and gauge invariant) terms such as $p_{ia} C_P^i D^{\text{aP}}$.

To summarise, for the generalised half-flat manifolds, characterised by the algebra (4.1), the Kähler potential is unchanged from the half-flat mirror case and still given by eq. (3.51), while the superpotential now reads

$$W = p_{i0} T^i + p_{ia} T^i Z^a - q_i^A T^i \mathcal{G}_A(Z) + 2\alpha' (p_{ia} - q_i^A \mathcal{G}_{Aa}) C_P^i D^{\text{aP}} \quad (4.9) \\ - \frac{\alpha'}{3} \left[\mathcal{K}_{ijk} \bar{j}_{\bar{P}\bar{R}\bar{S}} C^{i\bar{P}} C^{j\bar{R}} C^{k\bar{S}} + \tilde{\mathcal{K}}_{abc} j_{\text{PRS}} D^{\text{aP}} D^{\text{bR}} D^{\text{cS}} \right].$$

¹⁰It is not hard to check that for the generalised half-flat case the the tensor Υ^{\perp} takes a form similar to the one in the half-flat mirror case, eq. (3.14). This allows us again to absorb these pieces into a redefinition of the charged fields C and D .

5. Conclusions

In this paper, we have studied heterotic string compactifications at order α' on specific classes of manifolds with $SU(3)$ structure, namely half-flat mirror manifolds and their generalisations. These manifolds are related to certain Calabi-Yau manifolds which facilitates the explicit computation of the associated four-dimensional effective theories. In order to solve the Bianchi identity, we have employed the simplest possibility for the choice of the internal gauge bundle, a variant of the well-known standard embedding. The spin connection of half-flat manifolds has in general $SO(6)$ holonomy which suggests a low-energy gauge group $SO(10)$. However, we were able to absorb the pieces of the connection which would have been responsible for this breaking to $SO(10)$ into a vacuum redefinition of the matter fields. For the fields re-defined in this way, we found that the E_6 gauge symmetry is restored. The four-dimensional effective theory contains a dilaton s , $h^{1,1}$ Kähler moduli T^i and $h^{2,1}$ complex structure moduli Z^a , where $h^{1,1}$ and $h^{2,1}$ are the Hodge numbers of the associated Calabi-Yau manifolds. In addition, there are $h^{1,1}$ matter fields $C^{i\bar{P}}$ in the $\overline{\mathbf{27}}$ representation and $h^{2,1}$ matter fields D^{aP} in the $\mathbf{27}$ representation of E_6 . Hence, the low-energy field content is precisely the same as for analogous Calabi-Yau compactifications of the heterotic string. The half-flat manifolds are defined in the large radius and large complex structure limit and it is in this limit that the effective four-dimensional theory has been derived. For the Kähler potential we find

$$K = K_s(s, \bar{s}) + K_K(T, \bar{T}) + K_{cs}(Z, \bar{Z}) + \alpha' \left[4 \|\Omega\|^{-2/3} g_{ij} C^{i\bar{P}} \bar{C}^j_{\bar{P}} + \|\Omega\|^{2/3} g_{a\bar{b}} D^{aP} \bar{D}^{\bar{b}}_{\bar{P}} - 2(K_i K_a C^i_P D^{aP} + \text{c.c.}) \right], \quad (5.1)$$

where $\|\Omega\|^2 = \exp(K_K - K_{cs})$ and K_s , K_K and K_{cs} are the standard Calabi-Yau Kähler potentials for the dilaton, the Kähler moduli and the complex structure moduli. Explicit formulae for these Kähler potentials are given in eqs. (2.12), (A.17) and (A.22), respectively. The superpotential is given by

$$W = p_{i0} T^i + p_{ia} T^i Z^a - q_i^A T^i \mathcal{G}_A(Z) + 2\alpha' (p_{ia} - q_i^A \mathcal{G}_{Aa}) C^i_P D^{aP} - \frac{\alpha'}{3} \left[\mathcal{K}_{ijk} j_{\bar{P}\bar{R}\bar{S}} C^{i\bar{P}} C^{j\bar{R}} C^{k\bar{S}} + \tilde{\mathcal{K}}_{abc} j_{P\bar{R}\bar{S}} D^{aP} D^{b\bar{R}} D^{c\bar{S}} \right]. \quad (5.2)$$

where \mathcal{G}_A are the derivatives of the complex structure pre-potential \mathcal{G} , given in eq. (A.20), and κ_{ijk} and $\tilde{\kappa}_{abc}$ are the intersection numbers of the associated Calabi-Yau manifolds and its mirror. Further $j_{P\bar{R}\bar{S}}$ and $j_{\bar{P}\bar{R}\bar{S}}$ are the cubic E_6 invariant tensors. This expression for the superpotential is given for the generalised half-flat manifolds discussion in section 4. To specialise to half-flat mirror manifolds one should set $p_{i0} = e_i$, $p_{ia} = 0$ and $q_i^A = 0$. Note that in this case the CD mass term in W vanishes. NS-NS flux with flux parameters ϵ_A and μ^A leads to an additional superpotential of the usual form

$$W_{\text{flux}} = \epsilon_0 + \epsilon_a Z^a - \mu^A \mathcal{G}_A. \quad (5.3)$$

For completeness, we also mention that the gauge kinetic function f is given by the dilaton, $f = s$, at order α' , as expected for heterotic compactifications.

Let us now come back to some of the questions raised in the introduction. We have seen that the low-energy theory can be written in an E_6 invariant way due to a suitable choice of the gauge connection and a related definition of the matter fields C and D . With the effective theory at hand, we should now re-assess the question of what the low-energy gauge group actually is in the light of a possible spontaneous symmetry breaking in the effective theory. It is clear from the above results that for all choices of torsion and flux parameters, there exists a supersymmetric vacuum at $C = D = 0$ where the E_6 gauge group is, of course, unbroken. The existence of this vacuum means that our choice of gauge bundle was sensible and has provided a suitable background around which to consider fluctuations. When the CD term in the superpotential is present (that is, for generalised half-flat manifolds but not for half-flat mirror manifolds) there is also the possibility of a supersymmetric vacuum with $C, D \neq 0$ and of the order of the torsion parameters. The gauge symmetry in this vacuum is presumably broken to $SO(10)$ or even smaller. However, given that the torsion parameters are presumably quantised in string units the C and D VEVs would be rather large and it is doubtful if this vacuum can be considered as consistent in a theory derived as an expansion in the matter fields. We will investigate this question in detail in a forthcoming publication [37]. For the further discussion, let us focus on the E_6 preserving vacuum at $C = D = 0$. For generalised half-flat manifolds, when the term CD in the superpotential is present, some or all of the vector-like $\mathbf{27}, \overline{\mathbf{27}}$ pairs receive a large mass and will be removed from the low-energy spectrum. This is an explicit realisation of the usual lore by which vector-like pairs of matter fields are removed and, at low energy, one remains with a net number of families given by the Euler number $|\chi|/2 = |h^{1,1} - h^{2,1}|$. The above results open up various avenues for exploring the phenomenology of the heterotic string on manifolds with $SU(3)$ structure, including the question of heterotic moduli stabilisation [23], the precise nature of the family anti-family pairing and, if supersymmetry breaking vacua are found, the computation of soft masses and parameters. We hope to report on these issues in a future publication [37].

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A. Conventions and notations

In this appendix we present our conventions and formulae which we use throughout the paper.

A.1 General conventions

We denote real indices on the Calabi-Yau manifold by $m, n, \dots = 1, \dots, 6$, holomorphic ones by $\alpha, \beta, \dots = 1, 2, 3$ and anti-holomorphic ones by $\bar{\alpha}, \bar{\beta}, \dots = 1, 2, 3$. Tangent space indices are referred to with the same symbols as above, but with an additional tilde underneath, so for example we use \tilde{m} for a real tangent space index.

Indices $i, j, \dots = 1, \dots, h^{1,1}$ and $a, b, \dots = 1, \dots, h^{2,1}$ label objects on the moduli spaces of Kähler and complex structure deformations, respectively. We shall also use the capitalised versions of these indices to label projective coordinates on these spaces, that is, for example $A, B, \dots = 0, 1, \dots, h^{2,1}$ for the projective complex structure moduli space.

Finally we use capital letters from the middle of the alphabet M, N, \dots for the quantities which transform under **27** of E_6 .

Where possible we shall use form notation. We use the following conventions:

- We define a p -form as $F_p = \frac{1}{p!} F_{m_1 \dots m_p} dx^{m_1} \wedge \dots \wedge dx^{m_p}$.
- the exterior product of a p - and a q -form is defined as

$$F_p \wedge G_q = \frac{1}{p!q!} F_{m_1 \dots m_p} G_{m_{p+1} \dots m_{p+q}} dx^{m_1} \wedge \dots \wedge dx^{m_{p+q}}$$
 which implies the component relation $(F_p \wedge G_q)_{m_1 \dots m_{p+q}} = \frac{(p+q)!}{p!q!} F_{[m_1 \dots m_p} G_{m_{p+1} \dots m_{p+q}]}$, where the antisymmetrisation is always understood to be of unit norm.
- we define the Hogde star $*$ such that $F_p \wedge *F_p = \frac{1}{p!} F_{m_1 \dots m_p} F^{m_1 \dots m_p}$.

A.2 Conventions for Calabi-Yau manifolds

We now collect some equations and conventions in relation to Calabi-Yau moduli spaces. They will be applied identically to the $SU(3)$ structure manifolds in sections 3 and 4, which are the main subject of this paper.

We denote by J the Kähler form and by Ω the holomorphic $(3,0)$ form on the Calabi-Yau manifold X . We choose a basis (ω_i) of harmonic $(1,1)$ forms for the second cohomology group $H^{1,1}(X)$ and also introduce the dual $(2,2)$ forms $\tilde{\omega}^i$. Further, we require a symplectic basis (α_A, β^B) of the third cohomology. These forms satisfy the standard normalisation integrals

$$\int \omega_i \wedge \tilde{\omega}^j = \delta_i^j, \tag{A.1}$$

$$\int \alpha_A \wedge \beta^B = \delta_A^B, \quad \int \alpha_A \wedge \alpha_B = \int \beta^A \wedge \beta^B = 0. \tag{A.2}$$

The moduli space is parameterized by deformations of the Kähler form J and of the holomorphic $(3,0)$ form Ω which we expand as

$$J = v^i \omega_i, \tag{A.3}$$

$$\Omega = \mathcal{Z}^A \alpha_A - \mathcal{G}_A \beta^A. \tag{A.4}$$

Here, v^i denote the Kähler moduli and \mathcal{Z}^A are projective coordinates on the complex structure moduli space. Further, \mathcal{G}_A denote the first derivatives of the prepotential \mathcal{G} . The

complex structure moduli are given by

$$z^a = \frac{\mathcal{Z}^a}{\mathcal{Z}^0}, \quad (\text{A.5})$$

and, for convenience, we adopt the convention that $\mathcal{Z}^0 = 1$.

The metrics on these moduli spaces can be written as

$$g_{ij} = \frac{1}{4\mathcal{K}} \int \omega_i \wedge * \omega_j, \quad (\text{A.6})$$

$$g_{a\bar{b}} = -\frac{1}{\int \Omega \wedge \bar{\Omega}} \int \chi_a \wedge \bar{\chi}_{\bar{b}}, \quad (\text{A.7})$$

where χ_a form a basis for the $(2, 1)$ harmonic forms. Their relation to the above symplectic basis (α_A, β^B) is encoded in Kodaira's formula

$$\frac{\partial \Omega}{\partial z^a} = -K_a \Omega + \chi_a, \quad (\text{A.8})$$

where, K_a denotes the derivative of the complex structure Kähler potential, is given in (A.19). The inverse relations are somewhat more complicated to write down. They can be found in the literature, for example in appendix A of ref. [22] which follows the same conventions as the present paper. Here we shall only give the formulae for the $(1, 2)$ parts

$$\begin{aligned} (\beta^0)_{1,2} &= -\frac{1}{\int \Omega \wedge \bar{\Omega}} K_b g^{b\bar{a}} \bar{\chi}_{\bar{a}}, \\ (\beta^a)_{1,2} &= -\frac{1}{\int \Omega \wedge \bar{\Omega}} \left(g^{a\bar{b}} + z^a K_c g^{c\bar{b}} \right) \bar{\chi}_{\bar{b}}, \\ (\alpha_A)_{1,2} &= -\frac{1}{\int \Omega \wedge \bar{\Omega}} g^{a\bar{b}} (\mathcal{G}_{Aa} + K_a \mathcal{G}_A) \bar{\chi}_{\bar{b}}, \end{aligned} \quad (\text{A.9})$$

which are needed for various calculations throughout the paper. As these forms are real, the $(2, 1)$ parts can be simply obtained by complex conjugation.

Since Ω is a $(3, 0)$ form on a (almost) complex three-dimensional manifold, it should be proportional to the complex ϵ symbol. We write

$$\Omega_{\alpha\beta\gamma} = \|\Omega\| \epsilon_{\alpha\beta\gamma}, \quad (\text{A.10})$$

where the norm of Ω is defined as

$$\|\Omega\|^2 = \frac{1}{6} \Omega_{\alpha\beta\gamma} \bar{\Omega}^{\alpha\beta\gamma} = \frac{i}{\mathcal{K}} \int \Omega \wedge \bar{\Omega}, \quad (\text{A.11})$$

and \mathcal{K} denotes the volume of the Calabi-Yau manifold

$$\mathcal{K} = \frac{1}{6} \int J \wedge J \wedge J. \quad (\text{A.12})$$

Moreover, we use the conventions for the complex indices that Ω (and thus ϵ) is imaginary anti-self-dual (IASD)

$$*\Omega = -i\Omega, \quad (\text{A.13})$$

while the $(2, 1)$ forms χ are imaginary self-dual (ISD)

$$*\chi = i\chi . \quad (\text{A.14})$$

We will frequently use the isomorphism between the space $H^{2,1}(X)$ of $(2, 1)$ harmonic forms and the space $H^{0,1}(X, TX)$ of $(0, 1)$ harmonic forms with values in the holomorphic tangent bundle whose elements we denote by η_a . Explicitly, this isomorphism can be written as

$$(\eta_a)_{\bar{\alpha}}{}^{\alpha} = \frac{1}{2\|\Omega\|^2} (\chi_a)_{\beta\gamma}{}^{\alpha} \bar{\Omega}_{\bar{\alpha}}{}^{\beta\gamma} , \quad (\text{A.15})$$

In terms of the forms η , the metric on the moduli space of complex structure deformations can be expressed as

$$g_{a\bar{b}} = \frac{1}{\mathcal{K}} \int (\eta_a)^{\alpha\beta} (\bar{\eta}_{\bar{b}})_{\alpha\beta} . \quad (\text{A.16})$$

Here, as in the rest of the paper, we have suppressed the measure \sqrt{g} in the integral.

The Kähler deformations v^i can be viewed as imaginary parts of the complexified fields t^i . Written in terms of these complexified fields, the metric (A.6) is Kähler with associated Kähler potential

$$K_K(t) = -\ln \mathcal{K} = -\ln \left(\frac{1}{6} \mathcal{K}_{ijk} v^i v^j v^k \right) = -\ln \left[\frac{i}{48} \mathcal{K}_{ijk} (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k) \right] . \quad (\text{A.17})$$

Here we have used eq. (A.12) for the volume and the expansion (A.3) of J . The triple intersection numbers \mathcal{K}_{ijk} are given by

$$\mathcal{K}_{ijk} = \int \omega_i \wedge \omega_j \wedge \omega_k . \quad (\text{A.18})$$

Similarly, the complex structure moduli space metric (A.7) is also Kähler with associated Kähler potential

$$K_{cs}(z) = -\ln i \int \Omega \wedge \bar{\Omega} . \quad (\text{A.19})$$

This Kähler potential can be expressed explicitly in terms of the complex structure moduli and the prepotential \mathcal{G} using eq. (A.4). In the large complex structure limit the prepotential takes the form

$$\mathcal{G} = -\frac{1}{6} \frac{\tilde{\mathcal{K}}_{abc} z^a z^b z^c}{z^0} , \quad (\text{A.20})$$

with

$$\tilde{\mathcal{K}}_{abc} = -i \int (\eta_a)_{\bar{\alpha}}{}^{\alpha} (\eta_b)_{\bar{\beta}}{}^{\beta} (\eta_c)_{\bar{\gamma}}{}^{\gamma} \Omega_{\alpha\beta\gamma} \bar{\Omega}^{\bar{\alpha}\bar{\beta}\bar{\gamma}} . \quad (\text{A.21})$$

being — up to a constant normalisation — the intersection numbers of the Calabi-Yau manifold mirror to X . In this case, the complex structure Kähler potential K_{cs} is of the same form as the one for the Kähler moduli (A.17), that is,

$$K_{cs}(t) = -\ln \left[\frac{i}{48} \tilde{\mathcal{K}}_{abc} (z^a - \bar{z}^a)(z^b - \bar{z}^b)(z^c - \bar{z}^c) \right] . \quad (\text{A.22})$$

A.3 Commutators and traces

In this subsection we present our conventions for E_8 generators with respect to the maximal subgroup $SU(3) \times E_6$. We split the E_8 generators into four groups, $S_{\alpha\bar{\beta}}$, T^x , $T_{\alpha P}$ and $\bar{T}_{\bar{\alpha}\bar{P}}$, in line with the decomposition

$$(248) = \underbrace{(8, 1)}_{S_{\alpha\bar{\beta}}} \oplus \underbrace{(1, 78)}_{T^x} \oplus \underbrace{(3, 27)}_{T_{\alpha P}} \oplus \underbrace{(\bar{3}, \bar{27})}_{\bar{T}_{\bar{\alpha}\bar{P}}} . \quad (\text{A.23})$$

of the adjoint of E_8 under $SU(3) \times E_6$. Note that the index P is used to label objects which transform as a **27** under E_6 . The matrices $S_{\alpha\bar{\beta}}$ in the adjoint of $SU(3)$ are subject to the constraint $S_{\alpha}^{\alpha} = 0$.

With these conventions the E_8 commutation relations can be written as

$$[T_{\alpha P}, T_{\beta R}] = \epsilon_{\alpha\beta\bar{\gamma}} j_{\text{PR}}^{\bar{S}} \bar{T}_{\bar{\gamma}\bar{S}} , \quad (\text{A.24})$$

$$[T_{\alpha P}, \bar{T}_{\beta\bar{R}}] = g_{\alpha\bar{\beta}} k_{P\bar{R}}^x + g_{P\bar{R}} S_{\alpha\bar{\beta}} , \quad (\text{A.25})$$

$$[S_{\beta\bar{\gamma}}, T_{\alpha P}] = -g_{\alpha\gamma} T_{\beta P} + \frac{1}{3} g_{\beta\bar{\gamma}} T_{\alpha P} , \quad (\text{A.26})$$

$$[T_{\alpha P}, T^x] = -k_P^x{}^R T_{\alpha R} . \quad (\text{A.27})$$

Note that the j_{PRS} is the fully symmetric, cubic invariant of E_6 . One can easily show that $-k_P^x{}^R$ are the components of the E_6 generators in the **27** representation. The E_8 Jacobi identity implies that

$$j_{\text{PR}}^{\bar{S}} k_{S\bar{S}}^x + j_{\text{RS}}^{\bar{S}} k_{P\bar{S}}^x + j_{\text{SP}}^{\bar{S}} k_{R\bar{S}}^x = 0 . \quad (\text{A.28})$$

Finally we use the following normalisation for the **(3, 27)** generators

$$\text{tr}(T_{\alpha P} \bar{T}_{\beta\bar{P}}) = g_{\alpha\bar{\beta}} g_{P\bar{R}} . \quad (\text{A.29})$$

Combining the last equation with eq. (A.24) one finds the useful formula

$$\text{tr}(T_{\alpha P} T_{\beta R} T_{\gamma S}) = \epsilon_{\alpha\beta\bar{\gamma}} j_{\text{PRS}} . \quad (\text{A.30})$$

for the cubic E_6 invariants j_{PRS} .

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